Distinguishing Separable and Entangled States

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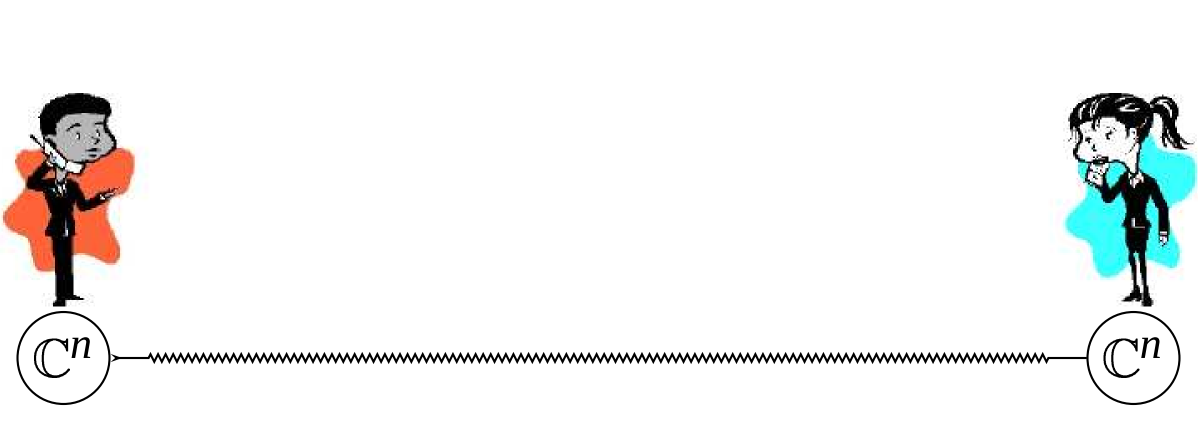

Entanglement of Mixed States

Composite quantum systems

- System A in ρ_A & System B in ρ_B

$\curvearrowright$ Composite system A $\otimes$ B in state $\rho_A \otimes \rho_B$.

- Mixing locally prepared states
  $\curvearrowright$ unentangled.

Definition

$\rho \in \mathrm{T}_1^+(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ is separable if

$$\rho = \sum_k \omega_k \rho_k^{(A)} \otimes \rho_k^{(B)}$$

where $\forall k$: $\omega_k \geq 0$, $\rho_k^{(A,B)} \in \mathrm{T}_1^+(\mathbb{C}^{d_{A,B}})$.

- in general difficult to decide $\curvearrowright$ separability criteria

Peres Criterion

Partial Transpose

$$\langle ij|\rho^{T_2}|kl\rangle \equiv \langle il|\rho|kj\rangle.$$

$$\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \rightarrow \begin{pmatrix} \rho_{11} & \rho_{21} & \rho_{13} & \rho_{23} \\ \rho_{12} & \rho_{22} & \rho_{14} & \rho_{24} \\ \rho_{31} & \rho_{41} & \rho_{33} & \rho_{43} \\ \rho_{32} & \rho_{42} & \rho_{34} & \rho_{44} \end{pmatrix}$$

- T is positive, but not completely positive.
- T is contractive, not completely contractive.

Peres PPT Criterion

If $\rho \in T(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ is separable, then

$$\rho^{T_2} \geq 0.$$

$\rho \in T(\mathbb{C}^2 \otimes \mathbb{C}^{2;3})$ is separable iff $\rho^{T_2} \geq 0$.

Positive maps approach

Theorem [Horodecki[3]]

ρ is separable iff $\mathbb{1}_A \otimes \Lambda_B(\rho) \geq 0$ for all positive linear $\Lambda_B : \mathrm{T}(\mathbb{C}^n) \to \mathrm{T}(\mathbb{C}^n)$.

Separability Criteria

If $\mathbb{1}_A \otimes \Lambda_B(\rho) \not\geq 0$, then ρ is entangled.

- Peres criterion: $\Lambda_B(\sigma) = \sigma^{\mathrm{T}}$.
- Reduction criterion: $\Lambda_B = \mathrm{tr}(\sigma)\mathbb{1} - \sigma$

Entanglement Witnesses

ρ is separable iff $\mathrm{tr}(W\rho) \geq 0$

for any operator W with $\mathrm{tr}(W\rho_1 \otimes \rho_2) \geq 0$.

- W with $\mathrm{tr}(W\rho) < 0$ is called an *entanglement witness* for ρ.

Bound entanglement

Distillability $\rho \overset{\text{LQCC}}{\Longrightarrow} |\psi_+\rangle\langle\psi_+|$.

Bound entangled states

- ρ entangled and $\rho \overset{\text{LQCC}}{\not\Rightarrow} |\psi_+\rangle\langle\psi_+|$.

Proposition ρ distillable $\Rightarrow$ $\rho^{T_2} \not\geq 0$.

Proof

✓ $\Lambda_{\text{LQCC}}(\sigma) = \sum_i A_i \otimes B_i \sigma A_i^\dagger \otimes B_i^\dagger$

✓ $(A \otimes B \rho A^\dagger \otimes B^\dagger)^{\mathrm{T}_2} = A \otimes B^* \rho^{\mathrm{T}_2} A^\dagger \otimes B^{\mathrm{T}}$.

↷ Λ_{LQCC} preserves PPT.

Cross Norms

Definition

Two normed spaces: $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$

$\curvearrowright$ Norm $\|\cdot\|_{AB}$ on $A \otimes_{\text{alg}} B$ is called

- cross if $\|\sigma_A \otimes \sigma_B\|_{AB} = \|\sigma_A\|_A \|\sigma_B\|_B$
- subcross if $\|\sigma_A \otimes \sigma_B\|_{AB} \leq \|\sigma_A\|_A \|\sigma_B\|_B$.

Quantum Information

$A, B = \mathtt{T}(\mathbb{C}^d), \|\cdot\|_{A,B} = \|\cdot\|_1$

$A \otimes_{\text{alg}} B = \mathtt{T}(\mathbb{C}^d \otimes \mathbb{C}^d) \cong d^2 \times d^2$ matrices.

✠

Trace norm: $\sigma \in \mathtt{T}(\mathbb{C}^d)$ with singular values $\lambda_1, \cdots, \lambda_d$.

$$\|\sigma\|_1 := \sum_{i=1}^{d} |\lambda_i|.$$

Hilbert-Schmidt norm: $\|\sigma\|_2 := \sqrt{\sum_{i=1}^{d} |\lambda_i|^2}.$

Greatest Cross Norm & Separability

The greatest cross norm

$$\|\sigma\|_\gamma := \inf\left\{ \sum_{i=1}^{k} \|\sigma_i^A\|_1 \|\sigma_i^B\|_1 \,\middle|\, \sigma = \sum_{i=1}^{k} \sigma_i^A \otimes \sigma_i^B \right\}$$

Theorem ρ is separable iff

- $\|\rho\|_\gamma = 1$.
- $\|\rho\|_{sc} \leq 1$ for all subcross norms $\|\cdot\|_{sc}$.

Greatest ? Let $\|\cdot\|_{sc}$ be a subcross norm

✓ $\|\sigma\|_{sc} \leq \sum_{i=1}^{k} \|\sigma_i^A\|_1 \|\sigma_i^B\|_1$ (subadditivity)

✓ take infimum on rhs. □

Computable Cross Norm

The computable subcross norm

$$\|\sigma\|_\tau := \inf\left\{\sum_{i=1}^k \|\sigma_i^A\|_2 \|\sigma_i^B\|_2 \,\middle|\, \sigma = \sum_{i=1}^k \sigma_i^A \otimes \sigma_i^B\right\}$$

- $\|\sigma\|_\tau \leq \|\sigma\|_\gamma \Rrightarrow \rho$ separable $\curvearrowright$ $\|\rho\|_\tau \leq 1$.

Computable ?

$$\|O\|_1 = \inf\left\{\sum_{i=1}^k \|\psi_i\|\|\phi_i\| \,\middle|\, O = \sum_{i=1}^k |\psi_i\rangle\langle\phi_i|\right\}$$

$\curvearrowright$ **Theorem** $\|\sigma\|_\tau = \|\mathfrak{A}(\sigma)\|_1$

- $\rho = \sum_i r_i E_i \otimes F_i \quad \longleftrightarrow \mathfrak{A}(\rho) = \sum_i r_i |E_i\rangle\langle F_i^*|$,

  $\rho \in \mathtt{T}(\mathbb{C}^d \otimes \mathbb{C}^d) \longleftrightarrow \mathfrak{A}(\rho) \in \mathtt{L}(\mathtt{T}(\mathbb{C}^d), \mathtt{T}(\mathbb{C}^d))$

- $$\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \leftrightarrow \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{21} & \rho_{22} \\ \rho_{13} & \rho_{14} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{41} & \rho_{42} \\ \rho_{33} & \rho_{34} & \rho_{43} & \rho_{44} \end{pmatrix}$$

CCN criterion and bound entanglement

Example $\mathbb{C}^3 \otimes \mathbb{C}^3$: $2 < \alpha \leq 5$,

$$\rho_\alpha := \frac{2}{7}\left|\Psi^+_{(3)}\right\rangle\left\langle\Psi^+_{(3)}\right| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-$$

$$\begin{aligned}\sigma_+ &\equiv \frac{1}{3}(|0\rangle|1\rangle\langle 0|\langle 1| + |1\rangle|2\rangle\langle 1|\langle 2| + |2\rangle|0\rangle\langle 2|\langle 0|)\\ \sigma_- &\equiv \frac{1}{3}(|1\rangle|0\rangle\langle 1|\langle 0| + |2\rangle|1\rangle\langle 2|\langle 1| + |0\rangle|2\rangle\langle 0|\langle 2|).\end{aligned}$$

$$\|\mathfrak{A}(\rho_\alpha)\|_1 = \frac{19}{21} + \frac{2}{21}\sqrt{19 - 15\alpha + 3\alpha^2}$$

↷ $\|\mathfrak{A}(\rho_\alpha)\|_1 > 1$, for $3 < \alpha \leq 5$.

- $\|\rho^{T_2}\|_1 \leq 1$ for $3 < \alpha \leq 4$.

↷ **ρ_α bound entangled for $3 < \alpha \leq 4$.**

Genuine Tripartite Entanglement

Three qubit state

$$\rho_{ABC} = \frac{1}{8}\left(\mathbb{1} - \sum_{i=1}^{4} |\psi_i\rangle\langle\psi_i|\right)$$

- $\{|\psi_i\rangle\} = \{|0,1,+\rangle, |1,+,0\rangle, |+,1,0\rangle, |-,-,-\rangle\}$
  with $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$
  = unextendable product basis.

- ρ_{ABC} is bi-separable and bound entangled.

- based on range analysis

  [Bennett et al. PRL **53** (1999), 5385].

Cross Norm Criterion

$$||(\mathbb{1} \otimes \mathfrak{A})\rho_{ABC}||_1 \simeq 1.08649$$

↷ ρ_{ABC} is entangled.

[source: M.,P., and R. Horodecki, quant-ph/0206008]

Linear contractions approach

Theorem

ρ is separable iff

- $\mathbb{1}_A \otimes \Lambda_B(\rho) \geq 0$
  for all positive trace-preserving linear
  $\Lambda_B : \mathrm{T}(\mathbb{C}^d) \to \mathrm{T}(\mathbb{C}^{2d})$.

- $\|\mathbb{1}_A \otimes \Lambda_B(\rho)\|_1 \leq 1$
  for all positive trace preserving linear
  $\Lambda_B : \mathrm{T}(\mathbb{C}^d) \to \mathrm{T}(\mathbb{C}^{2d})$.

- $\|\Lambda(\rho)\|_1 \leq 1$
  for all linear Λ with $\|\Lambda(\rho_1 \otimes \rho_2)\|_1 \leq 1$.

- $\|\rho\|_{sc} \leq 1$
  for all subcross norms $\|\cdot\|_{sc}$.