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BERRY PHASE FOR A SPIN ½ IN A CLASSICAL FLUCTUATING FIELD

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NEST
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Thanks to:

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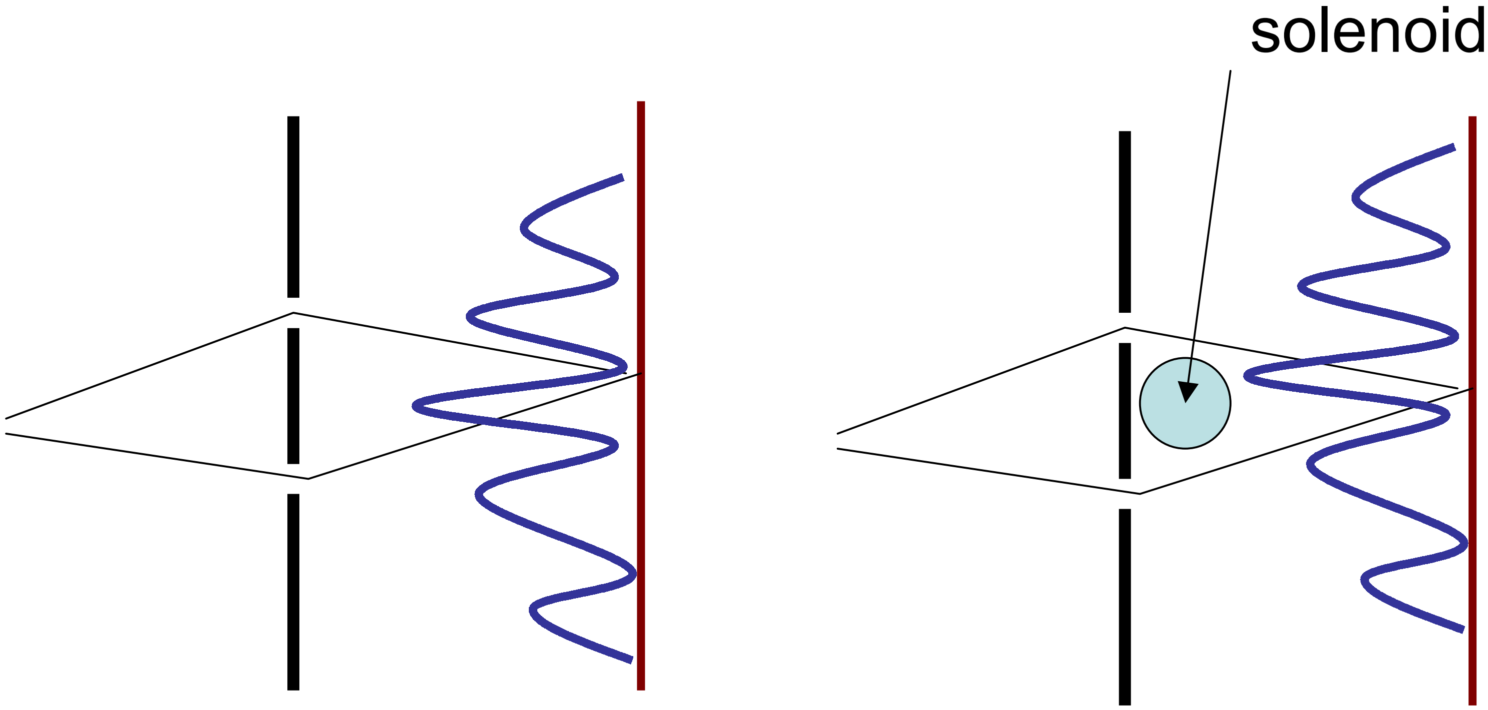
V.Vedral

OUTLINE

- *The A-B effect*
- *The adiabatic approximation*
- *The Berry phase*
- *Parallel transport*
- *Generalizations*
- *Conditional phase gates*
- *Classical noise in the evolution of spin ½*
- *Dynamic and geometric decoherence*

THE AHARONOV BOHM EFFECT: A GEOMETRIC EFFECT



In a double slit interference experiment with electrons the interference fringes are shifted by the presence of a localized magnetic field by an amount proportional to the magnetic flux across the solenoid.

Phase shift

$$\Delta\phi = \delta_0 + ie2\pi/hc\int_1 \mathbf{A}\cdot d\mathbf{s} - \int_2 \mathbf{A}\cdot d\mathbf{s}$$

$$= \delta_0 + ie2\pi/hc\int_{C12} \mathbf{A}\cdot d\mathbf{s}$$

$$= \delta_0 + ie2\pi/hc\iint_S \mathbf{B}\cdot d\mathbf{S}$$

$$= \delta_0 + i\Phi e2\pi/hc$$

Magnetic flux

- The electron never “touches” the magnetic field
- The fringe shift is gauge invariant
- The shift depends only on the path of the electrons

THE ADIABATIC APPROXIMATION

$$H(\boldsymbol{R}(t))\ |\Psi> = E(\boldsymbol{R}(t))\ |\Psi>$$

- The hamiltonian H depends on a set of control parameters $\boldsymbol{R}$
- The control parameters are changed adiabatically

If the system is initially in a non degenerate energy eigenstate it will remain in the corresponding eigenstate during the whole adiabatic evolution

THE BERRY PHASE

Assume that the change in the control parameters $\boldsymbol{R}$ is cyclic

$$\boldsymbol{R}(T) = \boldsymbol{R}(0) \quad \Longrightarrow \quad \begin{array}{l} E(T) = E(0) \\ H(T) = H(0) \end{array}$$

The final state differs from the initial one by a phase factor

$$|\Psi(T)> = \exp i\gamma \exp i\delta \, |\Psi(0)>$$

- Dinamical phase $\delta = \int_0^T E(t)dt$
- *Geometric (Berry) phase* γ

BERRY CONNECTION

$$\gamma = \int_C \boldsymbol{A} \cdot d\boldsymbol{R} \qquad \boldsymbol{A}(\boldsymbol{R})_n \equiv i < n(\boldsymbol{R}(t)) | \nabla_{\boldsymbol{R}} | n(\boldsymbol{R}(t)) >$$

BERRY PHASE

- The Berry connection is the analogue of the vector potential in the A-B effect
- It depends on the geometry of the trajectory in parameter space, e.g it is zero for a closed loop enclosing zero area

The Berry phase is invariant under gauge transformation and under path parameterization

$$| n(\boldsymbol{R}(t))> \longrightarrow | n'(\boldsymbol{R}(t))> = exp\; i\alpha(\boldsymbol{R})\, | n(\boldsymbol{R}(t))>$$

$$\boldsymbol{A}(\boldsymbol{R}) \longrightarrow \boldsymbol{A}'(\boldsymbol{R}) = \boldsymbol{A}(\boldsymbol{R}) - \nabla_{\boldsymbol{R}}\, \alpha(\boldsymbol{R})$$

$$\gamma' = \int_C \boldsymbol{A}' \cdot d\boldsymbol{R} = \int_C \boldsymbol{A} \cdot d\boldsymbol{R} + \int_C \nabla_{\boldsymbol{R}}\, \alpha(\boldsymbol{R}) \cdot d\boldsymbol{R} = \gamma$$

Gauge invariance is guaranted by cyclic evolution.

The Berry phase as a flux

$$\gamma = \int_C \boldsymbol{A} \cdot d\boldsymbol{R} = \iint_S \nabla \times \boldsymbol{A} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

- The Berry connection is a function only of the eigenstates dependence on the control parameters but it is independent from the energy spectrum.
- The Berry phase keeps a memory of the path followed in parameter space
- The dynamic phase keeps a memory on how fast the path is followed

AN EXAMPLE: A SPIN ½ IN A MAGNETIC FIELD

$$H = -\tfrac{1}{2}\,\boldsymbol{B}\cdot\boldsymbol{\sigma}$$

Eigenstates

$$|\uparrow>_B = e^{i\phi/2}\cos\theta/2\,|\uparrow>_z + e^{i\phi/2}\sin\theta/2\,|\downarrow>_z$$

$$|\downarrow>_B = e^{i\phi/2}\sin\theta/2\,|\uparrow>_z - e^{i\phi/2}\cos\theta/2\,|\downarrow>_z$$

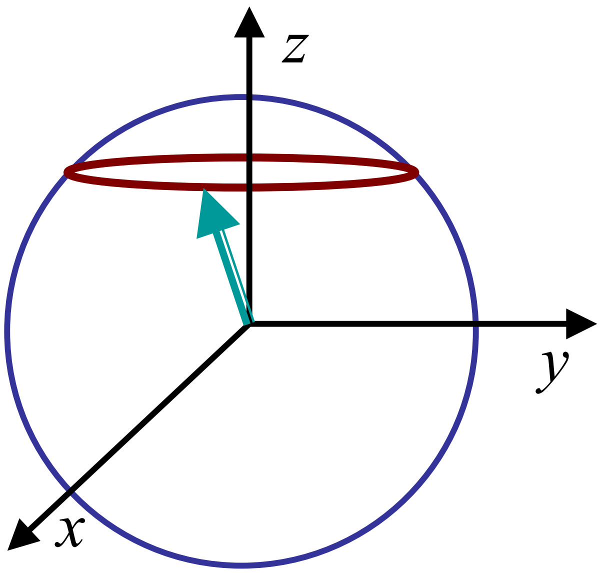
Independent from |B|

$$A_{\theta\uparrow} = A_{\theta\downarrow} = i<\uparrow_B|\partial/\partial\theta|\uparrow_B>= 0$$

$$A_{\phi\uparrow} = -A_{\phi\downarrow} = i<\uparrow_B|\partial/\partial\phi|\uparrow_B>= \tfrac{1}{2}\cos\theta$$

$$F_{\phi\theta\downarrow} = -F_{\phi\theta\uparrow} = \partial_\phi A_\theta - \partial_\theta A_\phi = \tfrac{1}{2}\sin\theta$$

AN EXAMPLE: PRECESSION AROUND A PARALLEL

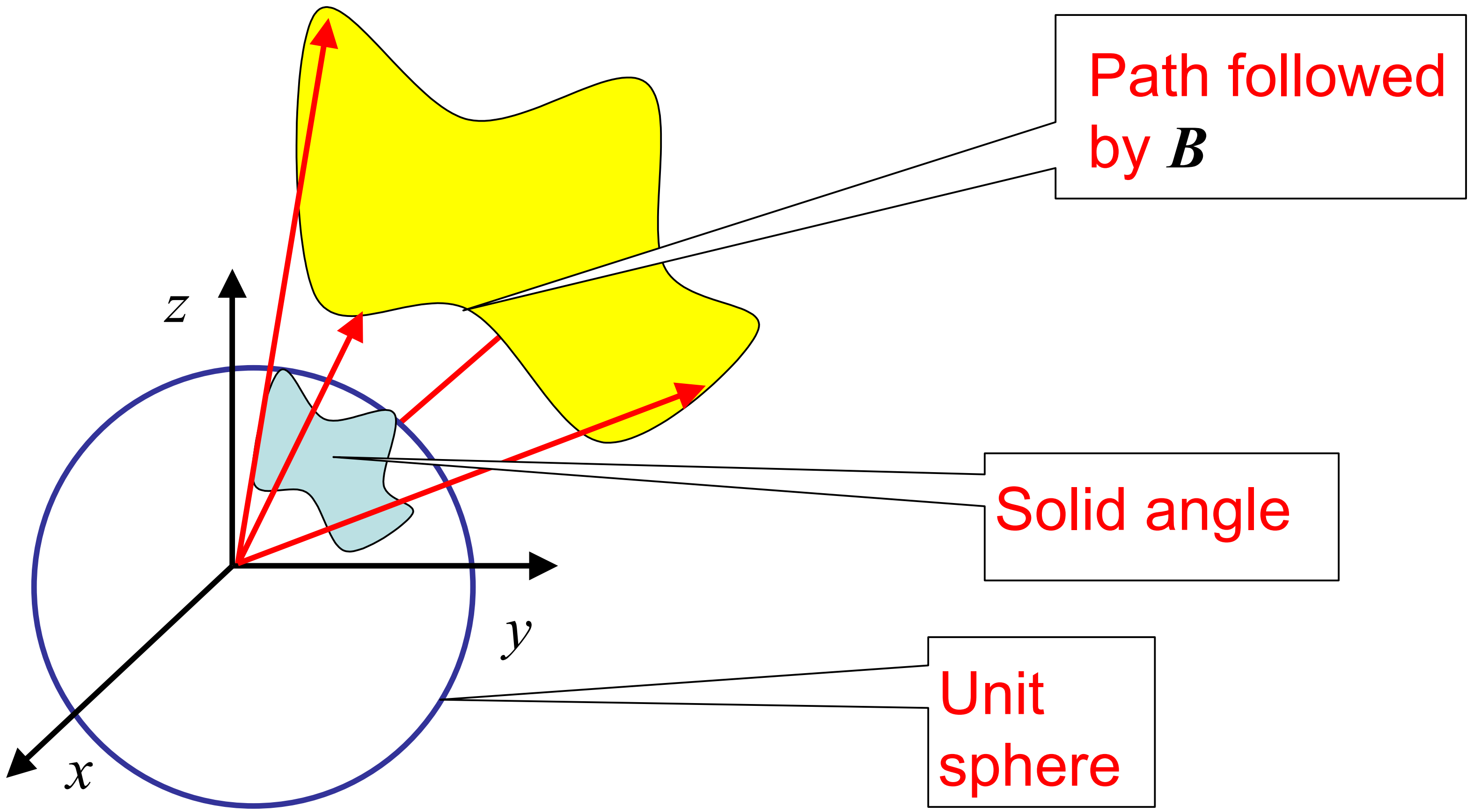


$\boldsymbol{B}$ precesses at an angle θ around the $\boldsymbol{z}$ axis with angular velocity Ω

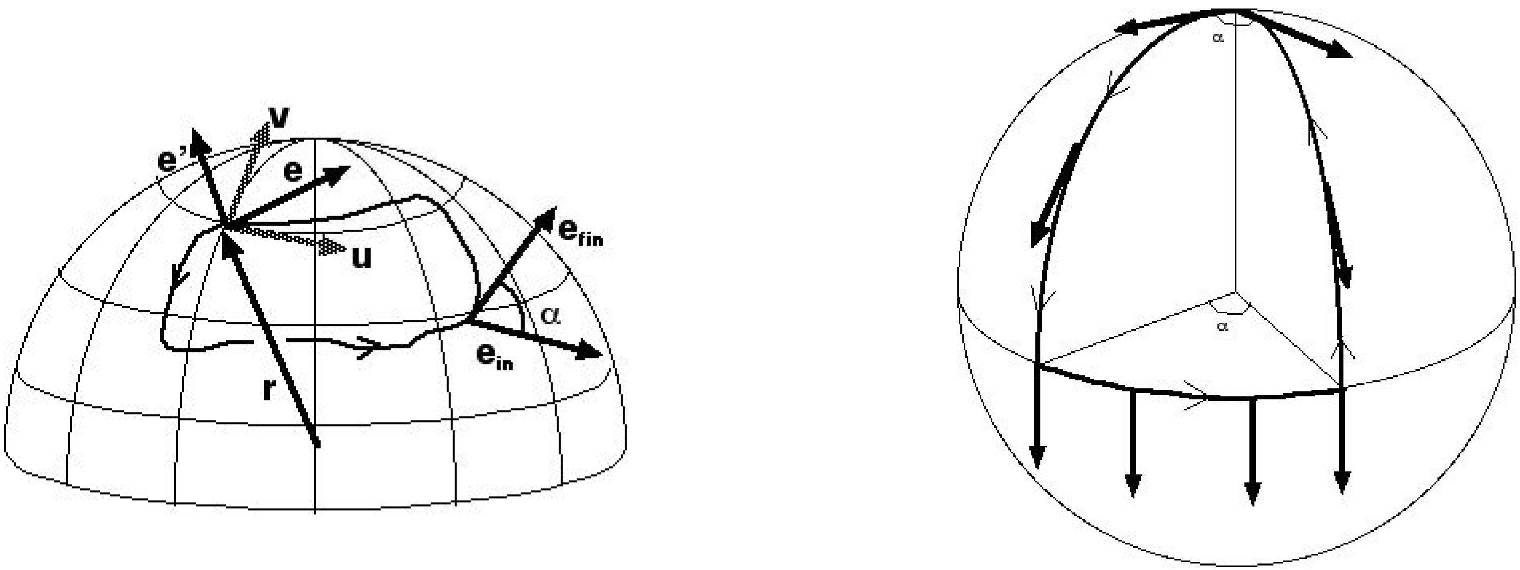
The Berry phase is independent from Ω

$$\gamma_{\uparrow} = -\gamma_{\downarrow} = -\int_0^{2\pi}\int_0^{\theta} \tfrac{1}{2} \sin\theta \, d\phi d\theta = -\pi\,(1 - \cos\theta)$$

The Berry phase is equal to the solid angle subtended by ***B*** at the degeneracy



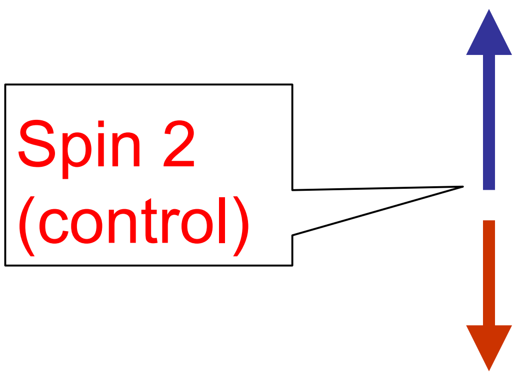
THE BERRY CONNECTION GIVES A RULE FOR THE PARALLEL TRANSPORT OF THE PHASE OF A QUANTUM STATE

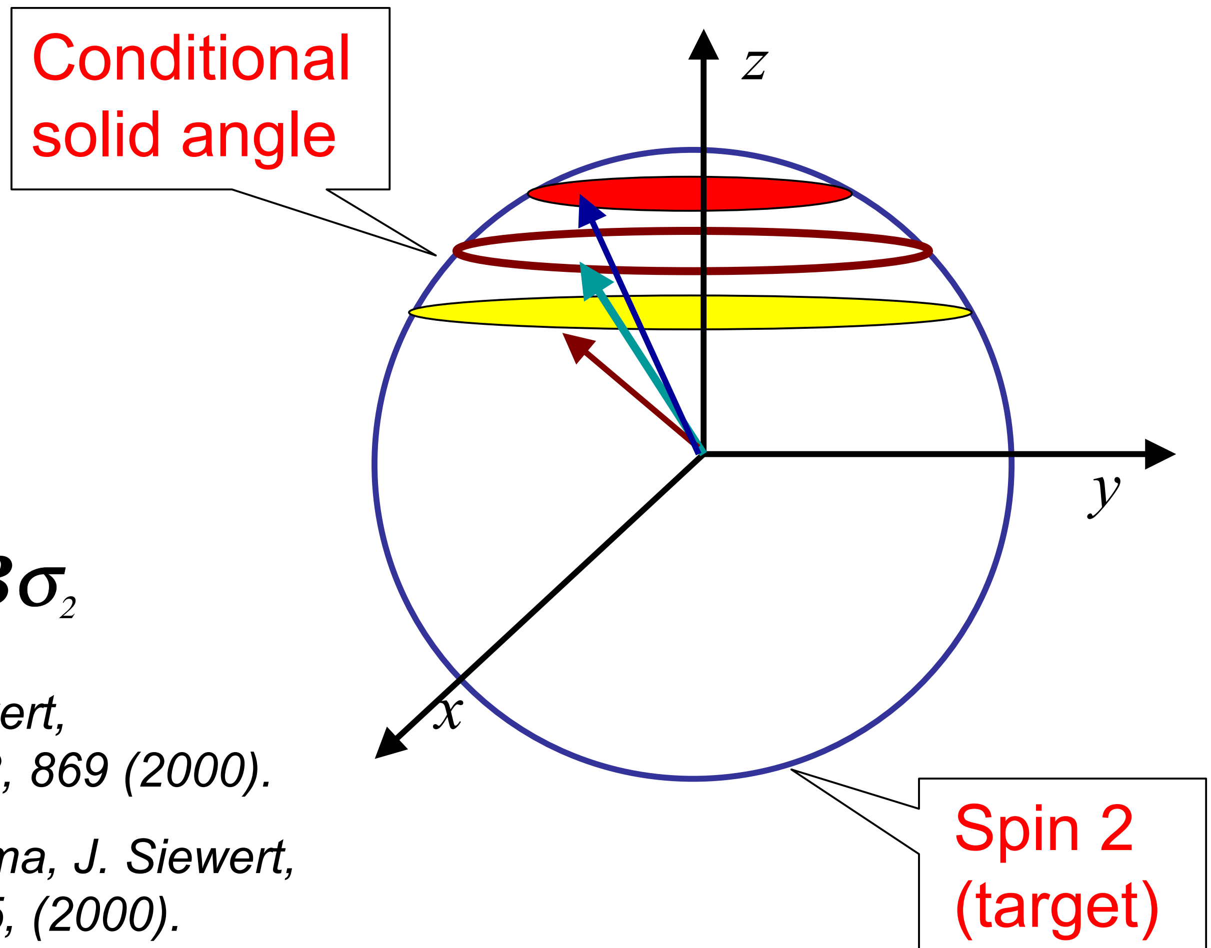


Parallel transport on a curved surface

CONDITIONAL PHASE SHIFT

$$H = + \tfrac{1}{2}\, \boldsymbol{B}_1 \cdot \boldsymbol{\sigma}_1 + \tfrac{1}{2}\, \boldsymbol{B}_2 \boldsymbol{\sigma}_{2\,+} + \Delta\, \boldsymbol{B} \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$$





$$\boldsymbol{B}_{1Eff} = \tfrac{1}{2}\, \boldsymbol{B}_1 + \Delta\, \boldsymbol{B} \boldsymbol{\sigma}_2$$

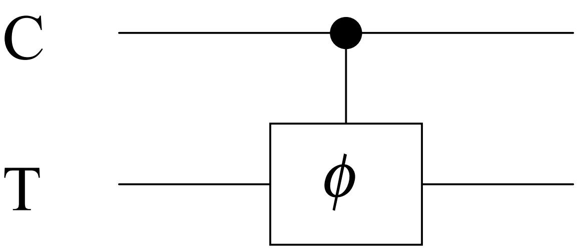
J. Jones, V. Vedral, A.K. Ekert,
C.Castagnoli, NATURE, 403, 869 (2000).

G. Falci, R. Fazio, G.M. Palma, J. Siewert,
V.Vedral, NATURE,407, 355, (2000).

GEOMETRIC QUANTUM PHASE GATE

The phase shift on the "target" spin (qubit) depends on the value of the "control" spin (qubit).

Can be used to implement a quantum conditional phase gate (a universal two-qubit gate)

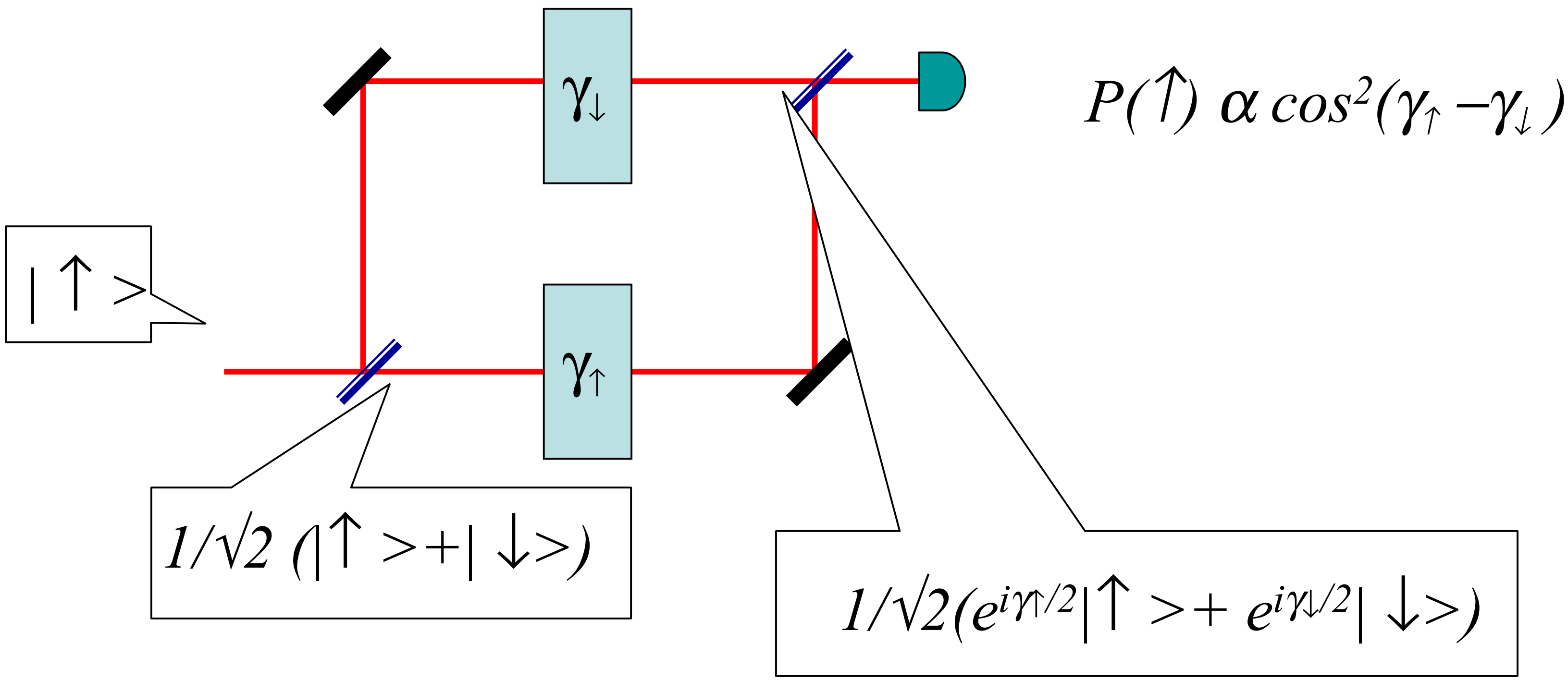


$$\begin{pmatrix} e^{i\gamma_{\uparrow}/2} & 0 & 0 & 0 \\ 0 & e^{-i\gamma_{\uparrow}/2} & 0 & 0 \\ 0 & 0 & e^{i\gamma_{\downarrow}/2} & 0 \\ 0 & 0 & 0 & e^{-i\gamma_{\downarrow}/2} \end{pmatrix}$$

With a suitable choice of path in parameter space it is possible to fix $\gamma_{\uparrow}$

HOW TO MEASURE THE BERRY PHASE

Mach – Zhender interferometer



•Superposition of energy eigenstates

•Cyclic adiabatic change of the Hamiltonian

•Mixing of the energy eigenstates

•Detection

$$<O> = \Sigma <i|O|j> \cos(\gamma_i - \gamma_j)$$

The last two steps amount to measuring an observable *O* which does not commute with the Hamiltonian

SPIN ECHO

It is possible to eliminate the dynamical contribution to the overall phase with a spin echo technique

$$|\Psi> = a|\uparrow> + b|\downarrow>$$

Cyclic evolution

$$|\Psi> = e^{i(\delta+\gamma)} a|\uparrow> + e^{-i(\delta+\gamma)} b|\downarrow>$$

Spin flip

$$|\Psi>_T = e^{i(\delta+\gamma)} a|\downarrow> + e^{-i(\delta+\gamma)} b|\uparrow>$$

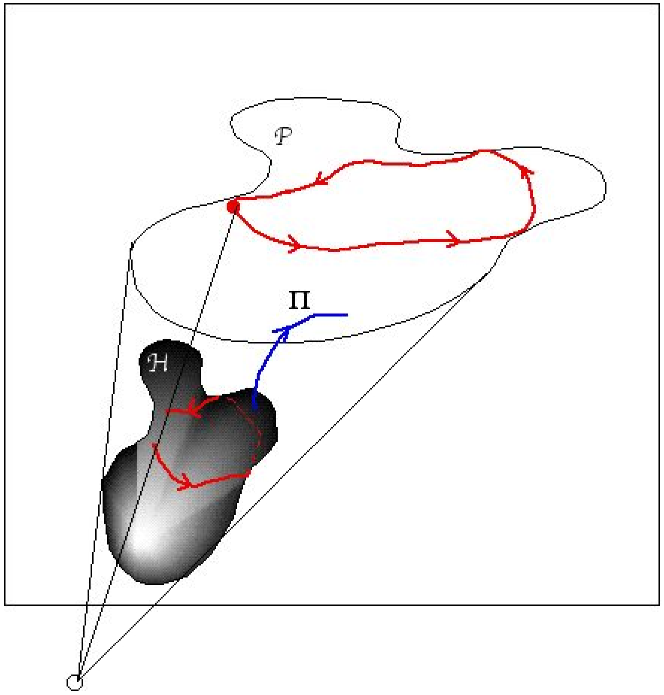
Cyclic evolution in the opposite verse

$$|\Psi>_T = e^{i2\gamma} a|\uparrow> + e^{-i2\gamma} b|\downarrow>$$

The dynamic phases cancel while the geometric one add

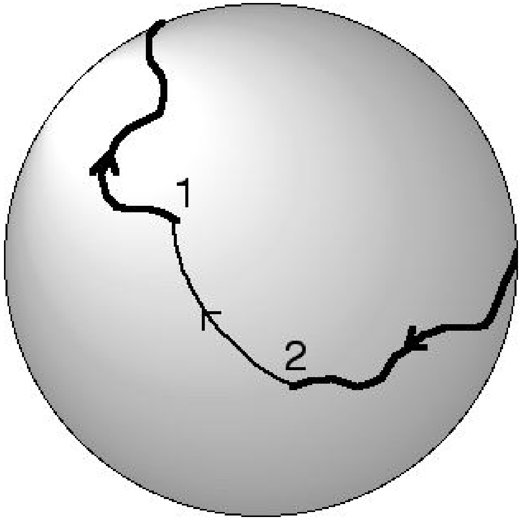
GENERALIZATION : THE AHARONOV ANANDAN PHASE

It is possible to relax the condition of adiabaticity. A geometric phase can be defined for cyclic evolution of a state in the projective Hilbert space



Such phase depends only on the closed trajectory in the projective Hilbert space

GENERALIZATION: NON CYCLIC EVOLUTION



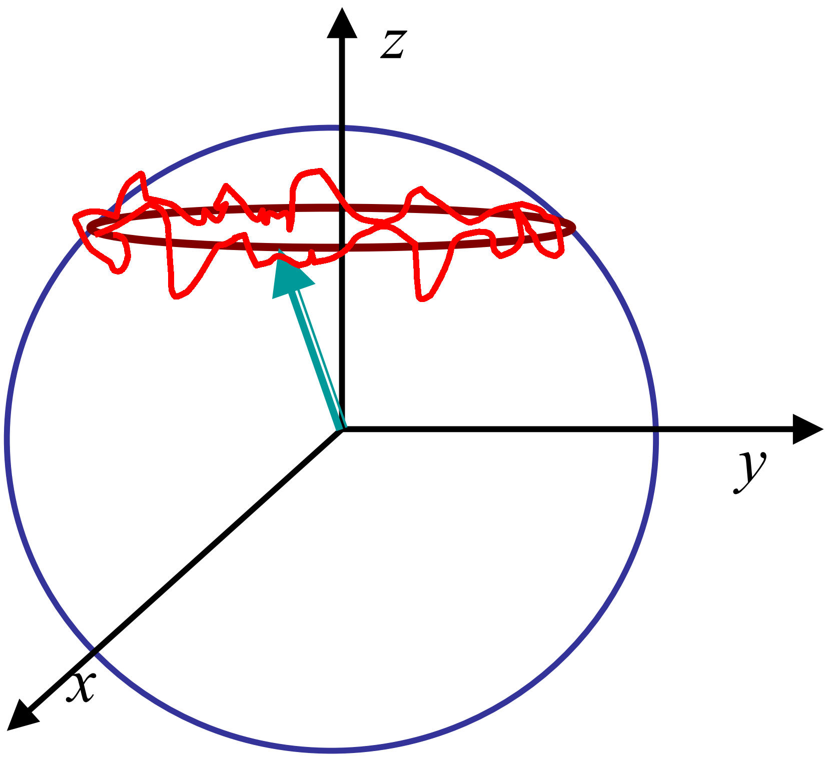
If the evolution is not cyclic it is still possible to define a geometric phase by closing the trajectory along a "geodetic" in parameter space

GENERALIZATION : NON ABELIAN PHASE

- *If the energy eigenspaces are non degenerate the effect of a cyclic adiabatic evolution is to unitarily mix them*
- *Such scheme is useful to implement all geometric quantum computation*

FAULT TOLERANT QUANTUM COMPUTATION

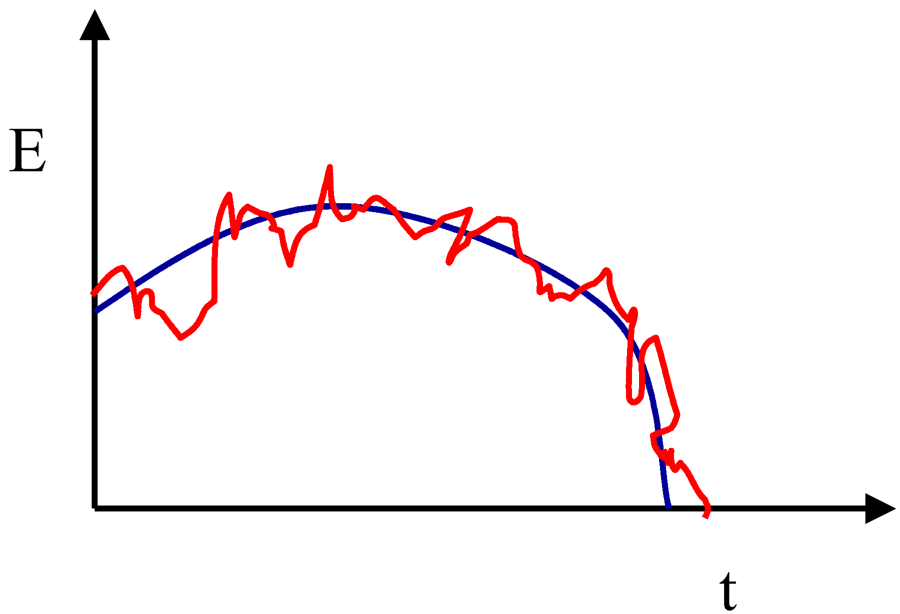
Geometric quantum computation is believed to be intrinsically more robust against random errors



As the geometric phase is proportional to the overall area traced on the unit sphere i.e. to a global property of the path in parameter space, errors with zero time average should not introduce errors

OBJECTION

•Dynamic phase $\delta = \int_0^T E(t)dt$



•The dynamical phase is proportional to the area of $E(t)$ vs. t

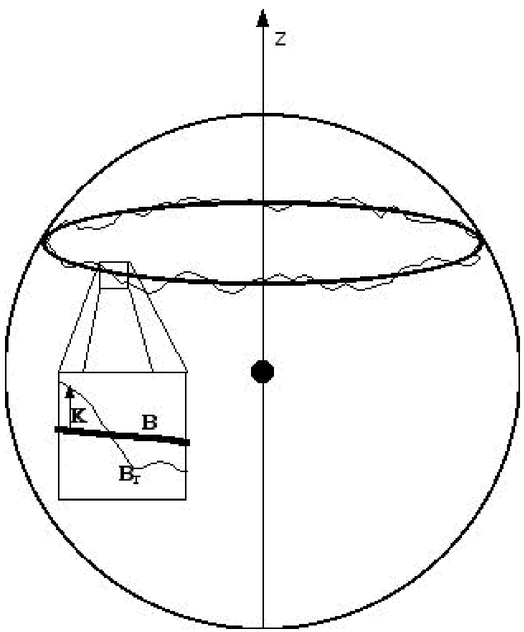
•Dynamic phase fluctuations are known to introduce decoherence

Do fluctuations play a different role in geometric and in dynamic phases?

THE NOISE MODEL

G.DeChiara G.M.Palma, Phys.Rev.Lett, in press

quan-ph/0303155



$$H = -\tfrac{1}{2}\, \boldsymbol{B}_T \cdot \boldsymbol{\sigma}$$

$$\boldsymbol{B}_T = \boldsymbol{B} + \boldsymbol{K}$$

Control field

Fluctuating field

A spin ½ interacting with a classical magnetic field with a small fluctuating component to model fluctuations in the control parameters

NOISE PROPERTIES

- $K \ll B$
- K is assumed to be a Ornstein –Uhlenbeck process with zero average and variance σ^2. It is therefore:
  - Gaussian
  - Markovian
  - Stationary
- The noise is “adiabatic”, i.e. the bandwidth of its Lorentian spectrum is smaller than B.

FIRST ORDER CORRECTIONS

First order correction the connection

$$A_\phi(\theta) \cong A_\phi(\theta_0) + \partial/\partial\theta\, A_\phi(\theta)\, \delta\theta$$
$$= ½\,(1 - \cos\theta_0 + \delta\theta \sin\theta_0)$$

For a precession around the z *axis* $\phi'_0 = 2\pi/T$

First order correction the line element

$$\delta\phi = \phi'\, dt \cong (\phi'_0 + \delta\phi')\, dt$$

First order correction to the Berry phase

$$\gamma = \int_0^T (A_\phi(\theta_0) + \delta A_\phi)(\phi'_0 + \delta\phi')\, dt$$

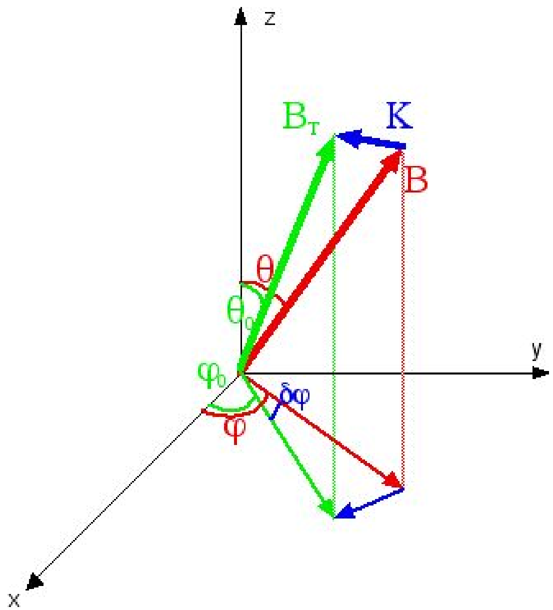
$$\cong \gamma_0 + 2\pi/T \int_0^T \delta A_\phi\, dt + A_\phi(\theta_0) \int_0^T \delta\phi'\, dt$$

$$\cong \gamma_0 + 2\pi/T \int_0^T \sin\theta_0 \delta\phi'\, dt + A_\phi(\theta_0)\, \delta\phi(T)$$

The connection fluctuates

The path does not close

APPROXIMATE EXPRESSIONS



$$\cos(\theta_0 + \delta\theta) \cong \cos\theta_0 - \delta\theta \sin\theta_0$$
$$= B_3/B + K_3/B - \boldsymbol{B}\cdot\boldsymbol{K}\, B_3/B^3$$

$$\gamma = \gamma_0 + 2\pi/T \int_0^T (K_3/B - \boldsymbol{B}\cdot\boldsymbol{K}\, B_3/B^3)dt$$

The Berry phase can be evaluated in terms of the cartesian component of $\boldsymbol{K}$, each of which is a gaussian random process with (generally different) variance.

It can be shown that the non cyclic corrections do not contribute.

ENERGY FLUCTUATIONS

The fluctuating field $\boldsymbol{K}$ introduces also fluctuations in the energy eigenvalues

$$H = -\tfrac{1}{2}\,(\boldsymbol{B} + \boldsymbol{K})\cdot\boldsymbol{\sigma}$$

To first order $\quad E = \pm\ 1/2\ (B + \boldsymbol{B}\cdot\boldsymbol{K}\,/\,B)$

$$\delta = \delta_0 + \int_0^T \boldsymbol{B}\cdot\boldsymbol{K}\,/\,B\ dt$$
$$= \delta_0 + \int_0^T \delta E(t)\,dt$$

DECOHERENCE

$|\Psi>_0 = a|\uparrow> \; + b\;|\downarrow>$ ϕ has a probability distribution $P(\phi)$

$|\Psi>_T = e^{i\phi}\, a|\uparrow> \; + e^{-i\phi}\, b|\downarrow>$

In our case the joint probability distribution for the dynamic + geometric phase is gaussian

$$\rho = \int |\Psi(\phi)>< \Psi(\phi)|\, d\phi \quad = \begin{pmatrix} |a|^2 & ab^* e^{i2\alpha} exp\{-2\sigma^2\} \\ a^*b\, e^{i2\alpha\uparrow} exp\{-2\sigma^2\} & |b|^2 \end{pmatrix}$$

Phase fluctuations generate decoherence

RESULTS

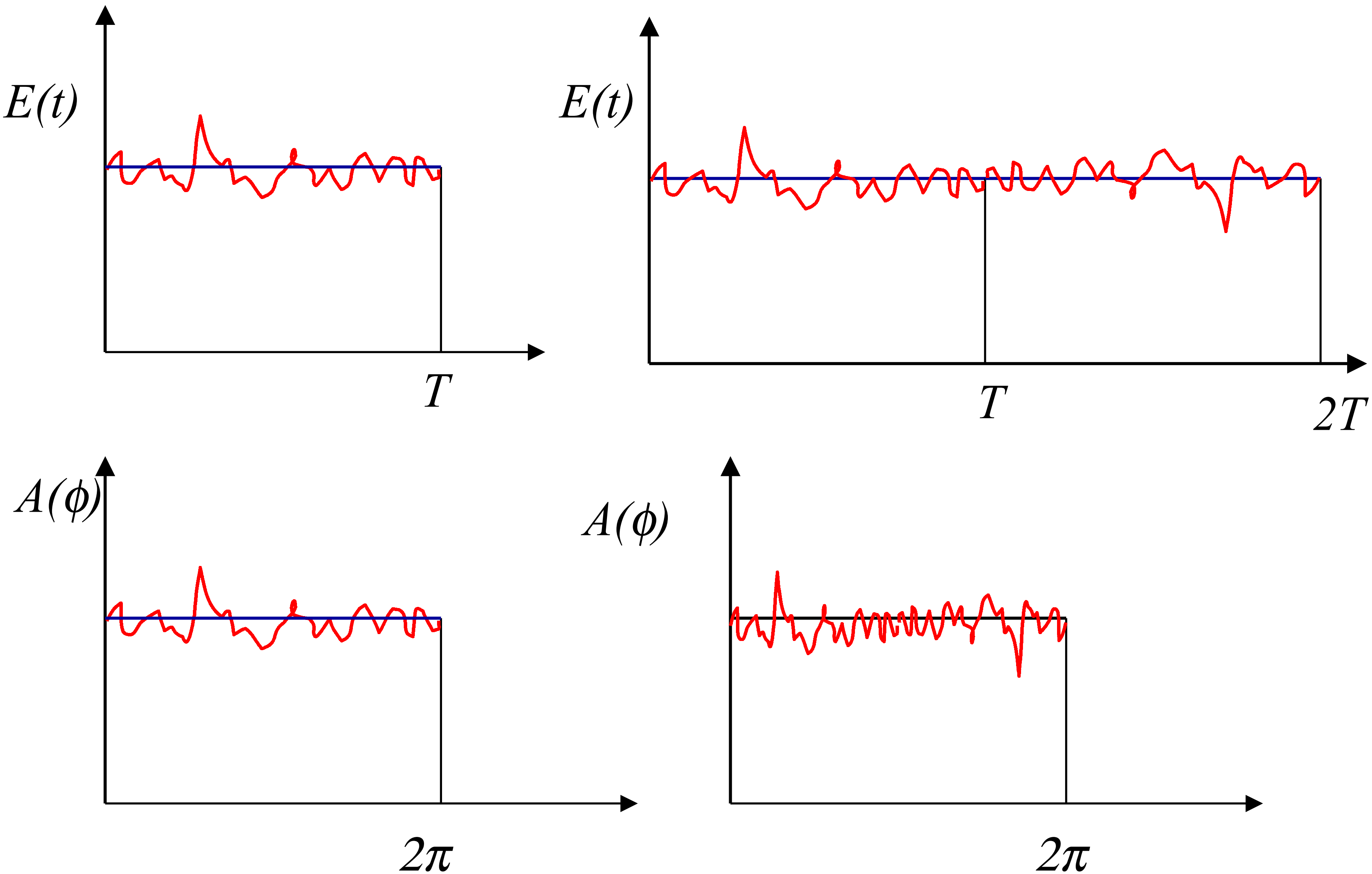
The variance grows linearly with time

In the adiabatic limit the variance is due to the dynamic contribution

The effect of noise is different in the geometric and in dynamic phases

$$\gamma \cong \gamma_0 + 2\pi/T \int_0^T \delta A_\phi \, dt$$

$$\delta \cong \delta_0 + \int_0^T \delta E(t) dt$$



E(t)
T
E(t)
T
2T
A(ϕ)
2π
A(ϕ)
2π

CONCLUSIONS

- *geometric phases are a general properties of quantum evolution*
- *can be used to implement quantum gates*
- *can generate entanglement*
- *geometric effects by themselves are more robust against noise*
- *the major source of decoherence are energy fluctuations.*