Estimation of squeezed state

ERATO, Quantum Computation and Information Project

&

COE, Graduate School of Information Science and Technology

Masahito Hayashi

Main purpose is finding optimal estimator for squeezed state:

and evaluating its performance.

Contents

- Properties of squeezed state
- Derivation of the optimal measurement with n copies
- Second order asymptotics
- Cloning

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Boson Fock space

Boson Fock space

a **: anihilation operator**

$a^\dagger$ **: creation operator**

$N \triangleq a^\dagger a$ **: number operator**

$|n\rangle_N$ **: number state (**$N|n\rangle_N = n|n\rangle_N$**),**

$|\alpha\rangle_a$ **:Boson coherent state (**$a|\alpha\rangle_a = \alpha|\alpha\rangle_a$**),**

Squeezed state

Caves's notation

$$\left|\alpha;\xi\right\rangle_c \triangleq \exp\left(-\frac{\xi}{2}\left(a^\dagger\right)^2+\frac{\bar{\xi}}{2}a^2\right)\left|\alpha\right\rangle_a$$

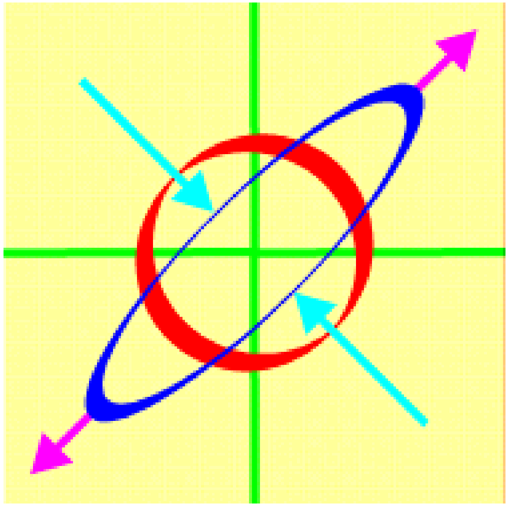
Yuen's notation

$$\left(\mu a+\nu a^\dagger\right)\left|\alpha;\mu,\nu\right\rangle_y=\alpha\left|\alpha;\mu,\nu\right\rangle_y,$$

where $\left|\mu\right|^2-\left|\nu\right|^2=1.$

$$\left|\alpha;\xi\right\rangle_c=\left|\alpha;\cosh\left|\xi\right|,e^{i\arg\xi}\sinh\left|\xi\right|\right\rangle_y.$$

Another Parameterization of Squeezed state



In the case of $\alpha=0$, we have

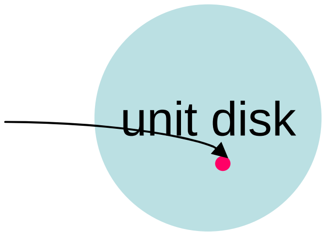
$$\left|0; e^{i\theta} t\right\rangle_c$$

$$= \left|0; \cosh t, e^{i\theta} \sinh t\right\rangle_y .$$

Parameterize

As another expresssion, we focus

on $\left|e^{i\theta} r\right\rangle_s \triangleq \left|0; e^{i\theta} \tanh^{-1} r\right\rangle_c ,$ $\quad e^{i\theta} r$



$e^{i\theta} r \in D$ **: unit disk.**

θ Angle of squeezing

r Scale of squeezing

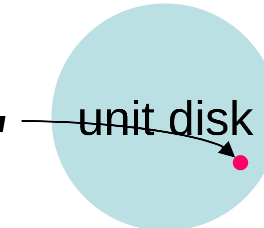
This is useful for group covariant method.

Merit of the expression $|\beta\rangle_s$

Since $\left(\mu a + \nu a^\dagger\right)|0;\mu,\nu\rangle_y = 0,$

$$-\left(a^\dagger\right)^{-1} a|0;\mu,\nu\rangle_y = \frac{\nu}{\mu}|0;\mu,\nu\rangle_y .$$

$e^{i\theta}r$



$\left(a^\dagger\right)^{-1}$ is defined as the inverse of

$a^\dagger : L^2_{\text{even}}(\mathbb{R}) \to L^2_{\text{odd}}(\mathbb{R}).$

Note that $|0;\mu,\nu\rangle_y \in L^2_{\text{even}}(\mathbb{R}).$

That is, $-\left(a^\dagger\right)^{-1} a|0;e^{i\theta}t\rangle_c = e^{i\theta}\tanh t|0;e^{i\theta}t\rangle_c .$

In other word, $-\left(a^\dagger\right)^{-1} a|\beta\rangle_s = \beta|\beta\rangle_s ,$

$|e^{i\theta}r\rangle_s = |0;e^{i\theta}\tanh^{-1} r\rangle_c , \; e^{i\theta}r \in D :$ unit disk.

n copies case of squeezed state

In coheret state case,

$$\frac{a_1 + \cdots a_n}{n} \left|\alpha\right\rangle_a^{\otimes n} = \alpha \left|\alpha\right\rangle_a^{\otimes n}.$$

In squeezed state case,

letting $a^{(n)} \triangleq -\left(\sum_{i=1}^{n}\left(a_i^{\dagger}\right)^2\right)^{-1}\sum_{i=1}^{n} a_i^{\dagger} a_i,$

we have $a^{(n)}\left|0;\mu,\nu\right\rangle_y^{\otimes n} = \frac{\nu}{\mu}\left|0;\mu,\nu\right\rangle_y^{\otimes n},$

$i.e.,$ $a^{(n)}\left|\beta\right\rangle_s^{\otimes n} = \beta\left|\beta\right\rangle_s^{\otimes n}.$

Proof of $a^{(n)}\left|0;\mu,\nu\right\rangle_y^{\otimes n} = \frac{\nu}{\mu}\left|0;\mu,\nu\right\rangle_y^{\otimes n}$

$$\left(\mu a_i + \nu a_i^{\dagger}\right)\left|0;\mu,\nu\right\rangle_y^{\otimes n} = 0,$$

$$i.e., -\mu a_i\left|0;\mu,\nu\right\rangle_y^{\otimes n} = \nu a_i^{\dagger}\left|0;\mu,\nu\right\rangle_y^{\otimes n}.$$

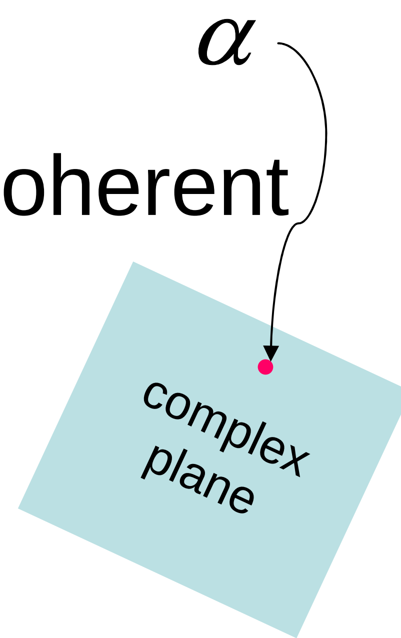
Thus, $-\mu a_i^{\dagger} a_i\left|0;\mu,\nu\right\rangle_y^{\otimes n} = \nu\left(a_i^{\dagger}\right)^2\left|0;\mu,\nu\right\rangle_y^{\otimes n}.$

Hence, $-\mu\sum_{i=1}^{n} a_i^{\dagger} a_i\left|0;\mu,\nu\right\rangle_y^{\otimes n} = \nu\sum_{i=1}^{n}\left(a_i^{\dagger}\right)^2\left|0;\mu,\nu\right\rangle_y^{\otimes n}.$

Therefore,

$$-\left(\sum_{i=1}^{n}\left(a_i^{\dagger}\right)^2\right)^{-1}\sum_{i=1}^{n} a_i^{\dagger} a_i\left|0;\mu,\nu\right\rangle_y^{\otimes n} = \frac{\nu}{\mu}\left|0;\mu,\nu\right\rangle_y^{\otimes n}.$$

Coherent state



- The annihilation operator $\boldsymbol{a}$ and coherent state $|\alpha\rangle_{\boldsymbol{a}}$ satisfy $\boldsymbol{a}|\alpha\rangle_{\boldsymbol{a}} = \alpha|\alpha\rangle_{\boldsymbol{a}}$.
- The heterodyne measurement $\boldsymbol{M}(\boldsymbol{d}\hat{\alpha}) \triangleq \frac{\mathbf{1}}{\mathbf{2}\pi}|\hat{\alpha}\rangle_{\boldsymbol{a}}\,{}_{\boldsymbol{a}}\langle\hat{\alpha}|\boldsymbol{d}^2\hat{\alpha}$ satisfies

$$\boldsymbol{a} = \int_{\mathbb{C}} \frac{\hat{\alpha}}{\mathbf{2}\pi}|\hat{\alpha}\rangle_{\boldsymbol{a}}\,{}_{\boldsymbol{a}}\langle\hat{\alpha}|\boldsymbol{d}^2\hat{\alpha},\ \boldsymbol{a}\boldsymbol{a}^\dagger = \int_{\mathbb{C}} \frac{|\hat{\alpha}|^2}{\mathbf{2}\pi}|\hat{\alpha}\rangle_{\boldsymbol{a}}\,{}_{\boldsymbol{a}}\langle\hat{\alpha}|\boldsymbol{d}^2\hat{\alpha}.$$

- The heterodyne measurement is the optimal estimator of the family $\left\{|\alpha\rangle_{\boldsymbol{a}}\,\middle|\,\alpha\in\mathbb{C}\right\}$.

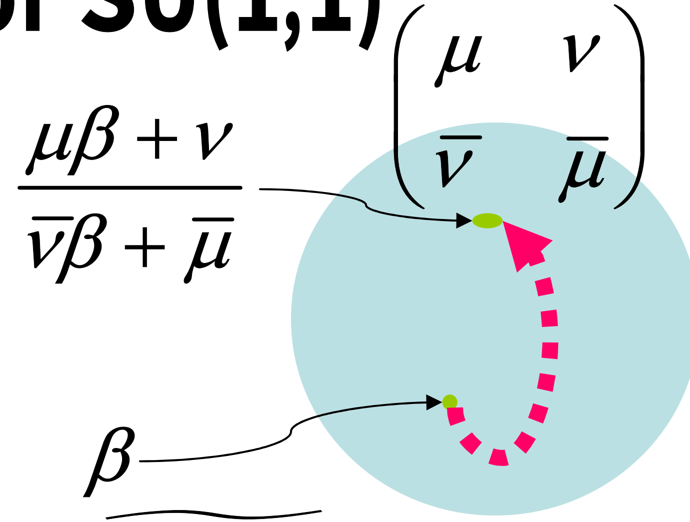
Similar properties are expected for squeezed state $|\beta\rangle_{\boldsymbol{s}}$ and operator $\boldsymbol{a}^{(1)} = -\left(\boldsymbol{a}^\dagger\right)^{-1}\boldsymbol{a}$ or $\boldsymbol{a}^{(n)}$.

The action of SU(1,1) : double-covering group of SU(1,1)

The representation V of $\widetilde{\mathrm{SU}(1,1)}$ is given as follows.

$$V(g)aV(g)^{\dagger} = \mu a + \nu a^{\dagger},$$

$\forall g \in \widetilde{\mathrm{SU}(1,1)}$, where



$\tilde{\pi}(g) = \begin{pmatrix} \mu & \nu \\ \bar{\nu} & \bar{\mu} \end{pmatrix}$ is the projection from $\widetilde{\mathrm{SU}(1,1)}$ to SU(1,1).

This representation acts on the squeezed state as

$$V(g)\left|\beta\right\rangle_{s\ s}\left\langle\beta\right|V(g)^{\dagger} = \left|\frac{\mu\beta+\nu}{\bar{\nu}\beta+\bar{\mu}}\right\rangle_{s\ s}\left\langle\frac{\mu\beta+\nu}{\bar{\nu}\beta+\bar{\mu}}\right|.$$

The invariant mearsure on the unit disk D is $\dfrac{d^2\beta}{\pi\left(1-\left|\beta\right|^2\right)^2}$.

Group covariace by $\widetilde{SU(1,1)}$ in the n - copy case

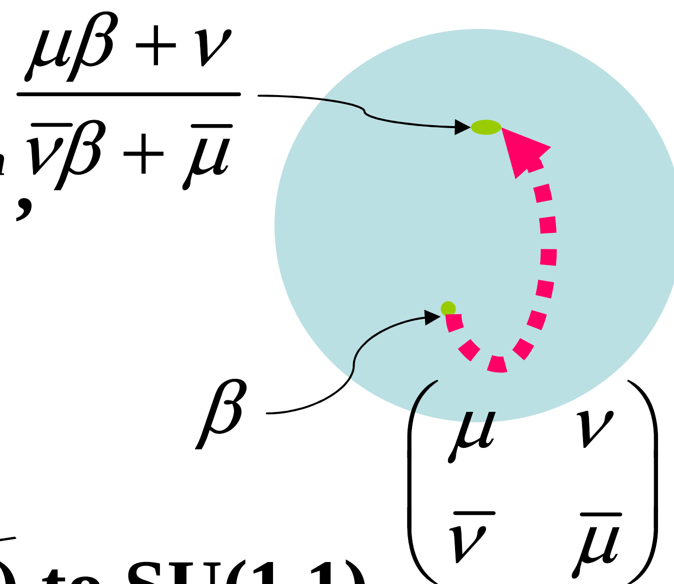
$$\rho_\beta \triangleq |\beta\rangle_{s\ s}\langle\beta|,\ \beta \in \boldsymbol{D} \triangleq \left\{ \boldsymbol{z} \in \mathbb{C} \,\middle\|\, |\boldsymbol{z}| < \boldsymbol{1} \right\},$$

We focus on the state family $\left\{ {\rho_\beta}^{\otimes n} \,\middle|\, \beta \in \boldsymbol{D} \right\}$

with the following action.

$$\boldsymbol{V}(\boldsymbol{g})^{\otimes n} {\rho_\beta}^{\otimes n} \left(\boldsymbol{V}(\boldsymbol{g})^{\otimes n} \right)^\dagger = {\rho_{\frac{\mu\beta+\nu}{\bar{\nu}\beta+\bar{\mu}}}}^{\otimes n},$$

where $\pi(\boldsymbol{g}) = \begin{pmatrix} \mu & \nu \\ \bar{\nu} & \bar{\mu} \end{pmatrix},$



π **is the projection from** $\widetilde{SU(1,1)}$ **to SU(1,1).**

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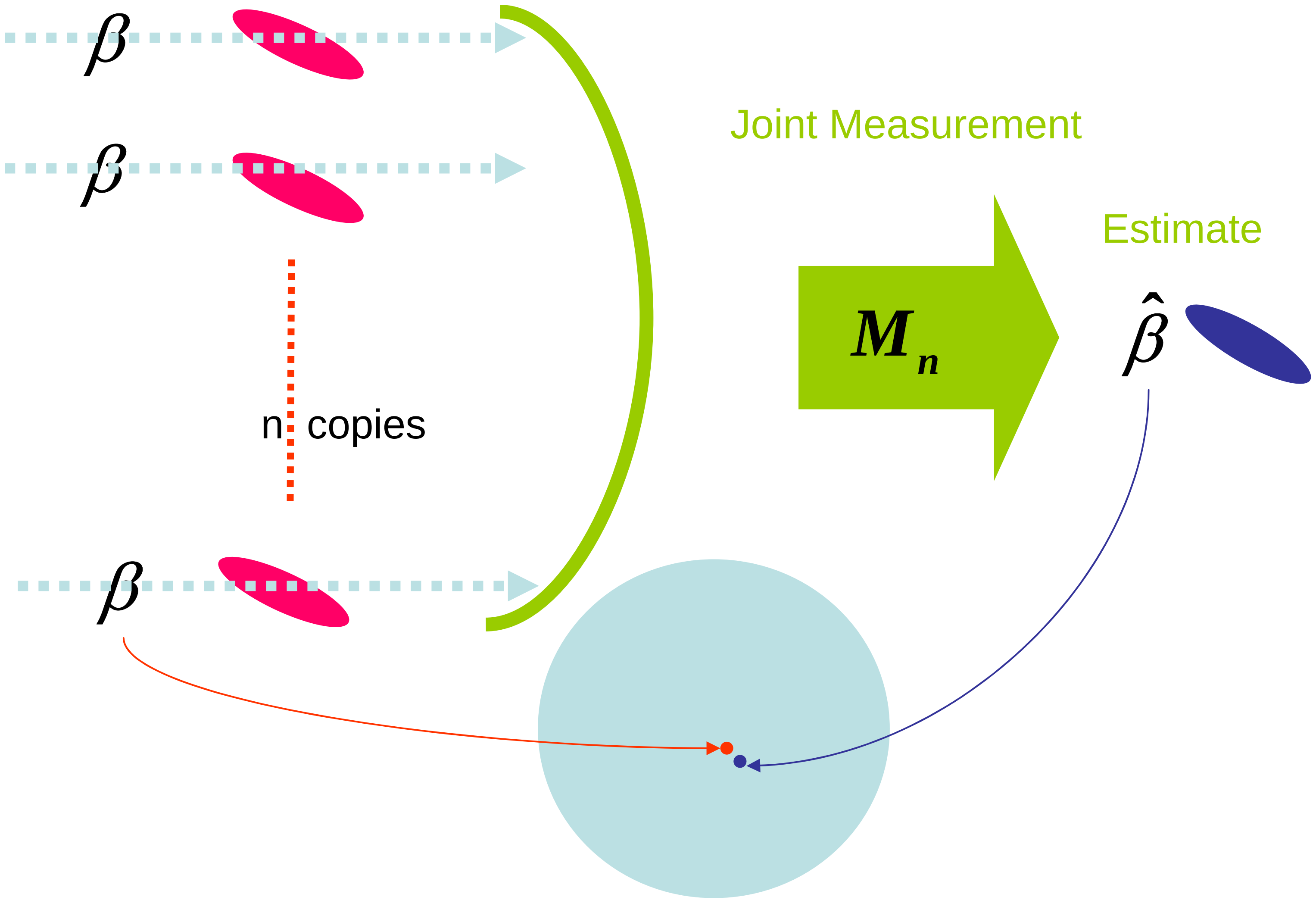
Optimal estimator



Optimal estimator

$n>2$, The error function $W(\beta,\hat{\beta})$ is a monotone decreasing function of the fidelity:

$$\left|{}_s\langle\beta|\hat{\beta}\rangle_s\right|^2=\left(1-\left|\frac{\beta-\hat{\beta}}{\bar{\beta}\hat{\beta}-1}\right|^2\right)^{\frac{1}{2}}.$$

The optimal measurement is

$$M_n(d^2\hat{\beta})\triangleq\left(\frac{n}{2}-1\right)\left(|\hat{\beta}\rangle_{s\,s}\langle\hat{\beta}|\right)^{\otimes n}\frac{d^2\hat{\beta}}{\pi(1-|\hat{\beta}|^2)^2}.$$

Its optimal distribution is

$$\mathrm{Tr}M_n(d^2\hat{\beta})\rho_\beta^{\otimes n}=\left(\frac{n}{2}-1\right)\left(1-\left|\frac{\beta-\hat{\beta}}{\bar{\beta}\hat{\beta}-1}\right|^2\right)^{\frac{n}{2}}\frac{d^2\hat{\beta}}{\pi(1-|\hat{\beta}|^2)^2}.$$

Framework of group covariant estimation

Assume that the state family $\left\{ \rho_\theta \middle| \theta \in \Theta \right\}$ **and group representation** V **of the grpup** G **satisfy that** $V(g)\rho_\theta V(g)^\dagger = \rho_{\pi(g)\theta}$, **where** π **is the action of** G **to** Θ.

Suppose that the error function $W(\theta,\hat{\theta})$ **satisfies**
$W(\theta,\hat{\theta}) = W(\pi(g)\theta,\pi(g)\hat{\theta})$, *e.g.*, $1-$**fidelity** *etc*.

$$D_\theta^W(M) \triangleq \int_\Theta W(\theta,\hat{\theta})\mathrm{Tr} M(d\hat{\theta})\rho_\theta .$$

Minimax method: minimize $D^W(M) \triangleq \sup_{\theta\in\Theta} D_\theta^W(M)$.

Quantum Hunt-Stein's lemma:

$$\min_M D^W(M) = \min_{M:\mathrm{cov}} D_\theta^W(M).$$

M **is covariant if** $M(\pi(g)d\hat{\theta}) = V(g)M(d\hat{\theta})V(g)^\dagger$.

Optimal performance

Fidelity:

$$\int_D \left| {}_s\langle \beta | \hat{\beta} \rangle_s \right|^2 \mathrm{Tr} M_n(d^2\hat{\beta}) \rho_\beta^{\otimes n}$$

$$= 1 - \frac{1}{n-1} \approx 1 - \frac{1}{n} + (1-2)\frac{1}{n^2}.$$

Square of Bures' distance:

$$\int_D \left(1 - \left| {}_s\langle \beta | \hat{\beta} \rangle_s \right| \right) \mathrm{Tr} M_n(d^2\hat{\beta}) \rho_\beta^{\otimes n}$$

$$= \frac{2}{2n-3} \approx \frac{1}{n} - \left(\frac{1}{2} - 2 \right) \frac{1}{n^2}.$$

Relation to the operator $a^{(n)}$

The measurement

$$M_n(d^2\beta) \triangleq \left(\frac{n}{2}-1\right)\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\frac{d^2\beta}{\pi(1-|\beta|^2)^2}$$

and the opeartor $a^{(n)} \triangleq -\left(\sum_{i=1}^{n}\left(a_i^{\dagger}\right)^2\right)^{-1}\sum_{i=1}^{n}a_i^{\dagger}a_i$

satisfiy the property similar to coherent case, $i.e.$**,**

$$a^{(n)} = \int_D \beta M_n(d^2\beta),\ a^{(n)}a^{(n)\dagger} = \int_D |\beta|^2 M_n(d^2\beta).$$

$$\because \quad a^{(n)} = \int_D a^{(n)} M_n(d^2\beta) = \int_D \beta M_n(d^2\beta).$$

$$a^{(n)}a^{(n)\dagger} = \int_D a^{(n)} M_n(d^2\beta) a^{(n)\dagger}$$

$$= \int_D \beta M_n(d^2\beta)\overline{\beta} = \int_D |\beta|^2 M_n(d^2\beta).$$

Case of n=1, n=2

In the case of $n=1,2$, **c.f. Optimal POVM**

$$\int_D \left(|\beta\rangle_{s\ s}\langle\beta| \right)^{\otimes n} \frac{d^2\beta}{\pi(1-|\beta|^2)^2} = \infty \cdot \left(|\beta\rangle_{s\ s}\langle\beta| \right)^{\otimes n} \frac{\left(\frac{n}{2}-1\right) d^2\beta}{\pi(1-|\beta|^2)^2}$$

Hence, there is no optimal covariant measurement.

However, in the case of $n=2$, there exsists

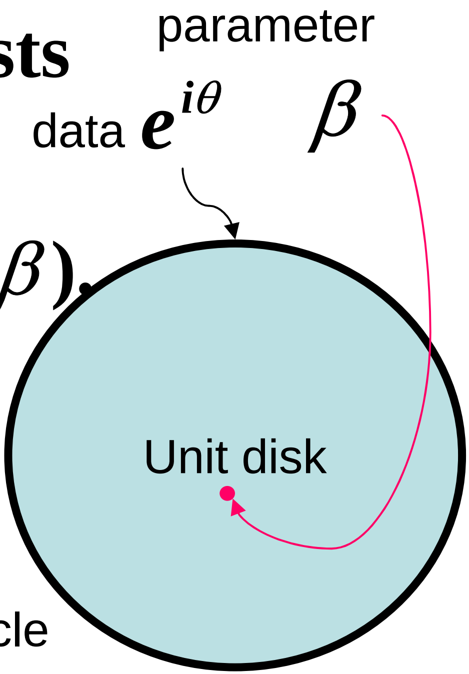
a POVM M_2 such that

$$a^{(2)} = \int_U \beta M_2(d\beta),\ a^{(2)} a^{(2)\dagger} = \int_U |\beta|^2 M_2(d\beta),$$

the POVM M_2 has the measuring data

in the unit circle $U \triangleq \left\{ z \in \mathbb{C} \,\middle\|\, z| = 1 \right\}$.

Note that it is out of parameter space.



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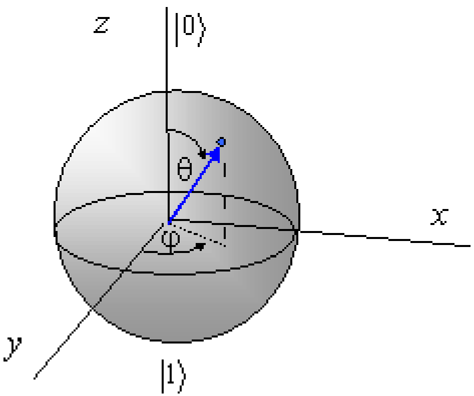
Relation to scalar curvature

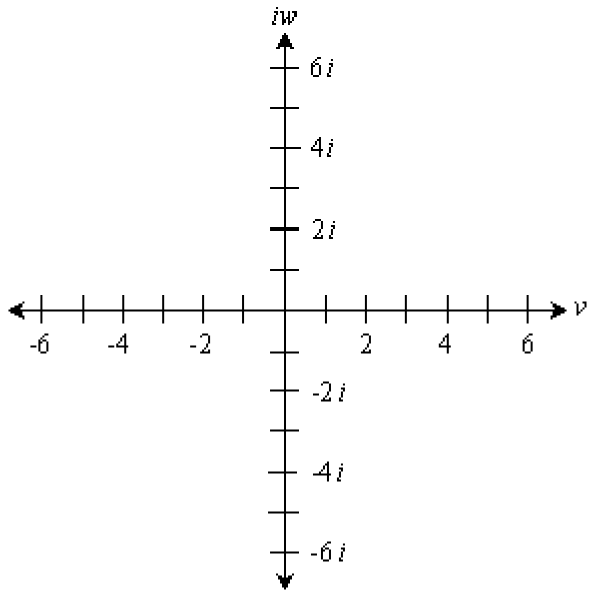
Qubit state family

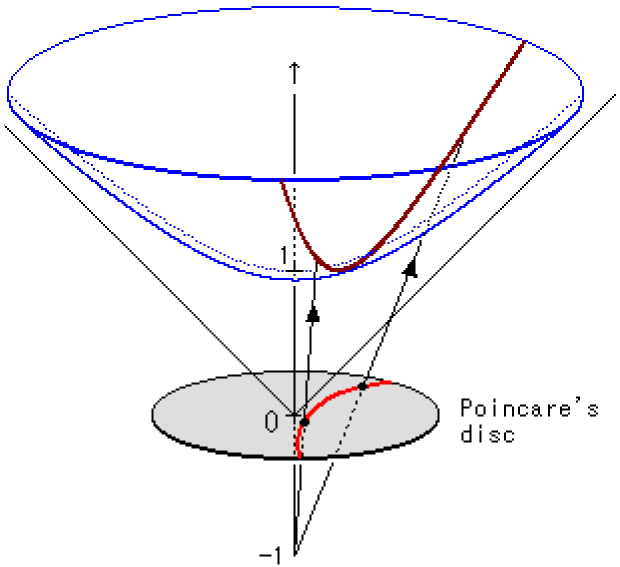
Coherent state family

Squeezed state family



$$w = w_0|0\rangle + w_1|1\rangle \equiv \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$





Bloch sphere

Complex plane

Unit disk=Hyperboloid

Scalar curvature 2

Scalar curvature 0

Scalar curvature -4

Thanks to http://www1.kcn.ne.jp/~iittoo/index.htm

Optimal performance with the second order asymptotics

Fidelity | **Square of Bures' dis.**

Qubit: $\frac{n+1}{n+2} \approx 1 - \frac{1}{n} + (1 + 1)\frac{1}{n^2}$, $\frac{2}{2n+3} \approx \frac{1}{n} - \left(\frac{1}{2} + 1\right)\frac{1}{n^2}$

coherent: $\frac{n}{n+1} \approx 1 - \frac{1}{n} + (1 + 0)\frac{1}{n^2}$, $\frac{2n}{2n+1} \approx \frac{1}{n} - \left(\frac{1}{2} + 0\right)\frac{1}{n^2}$

Squeezed: $\frac{n-2}{n-1} \approx 1 - \frac{1}{n} + (1 - 2)\frac{1}{n^2}$, $\frac{2}{2n-3} \approx \frac{1}{n} - \left(\frac{1}{2} - 2\right)\frac{1}{n^2}$

Scalar curvature

Qubit state family: $2 = 2 \times 1$

Coherent state family: $0 = 2 \times 0$

Squeezed state family: $-4 = 2 \times -2$

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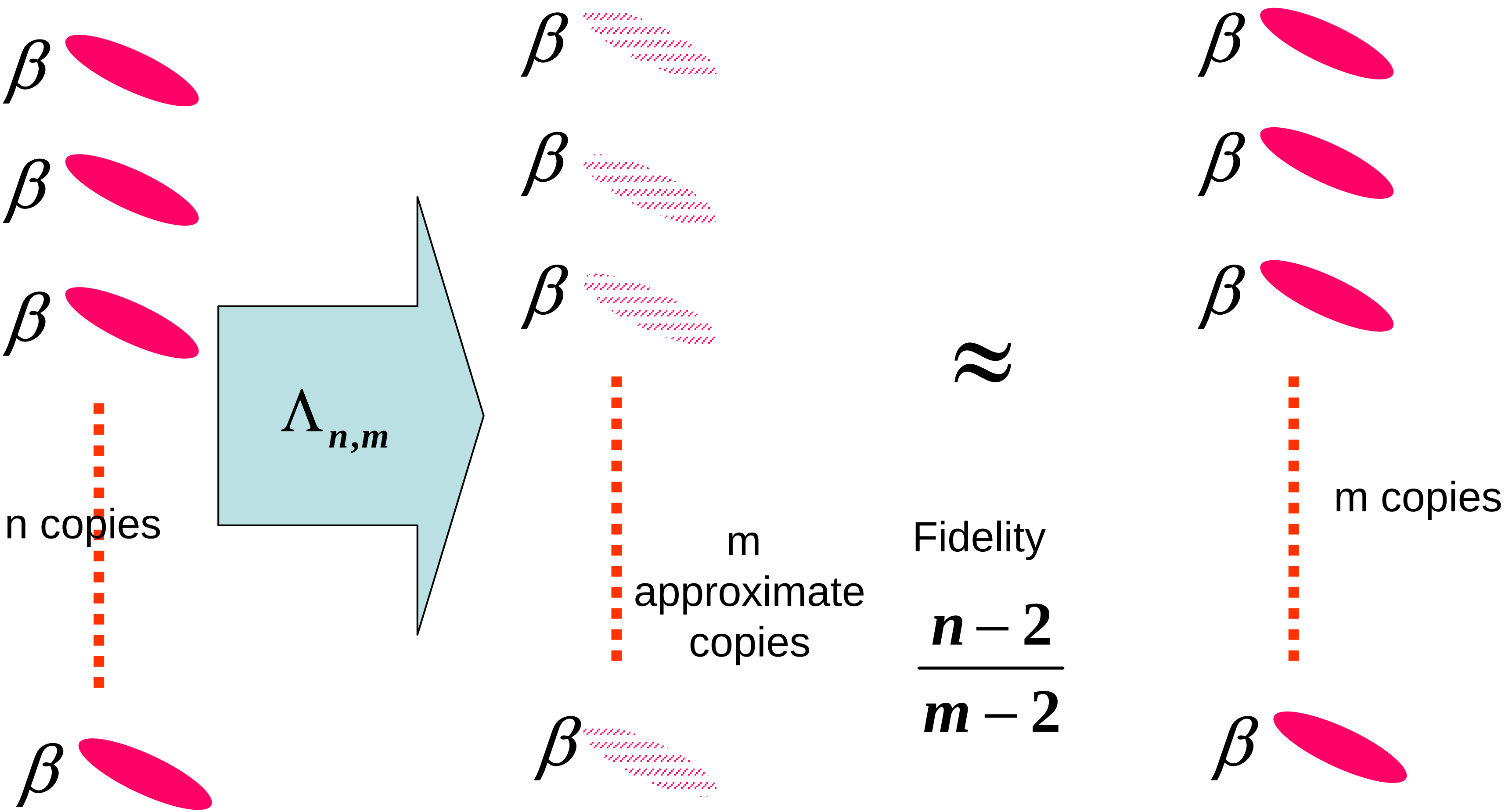
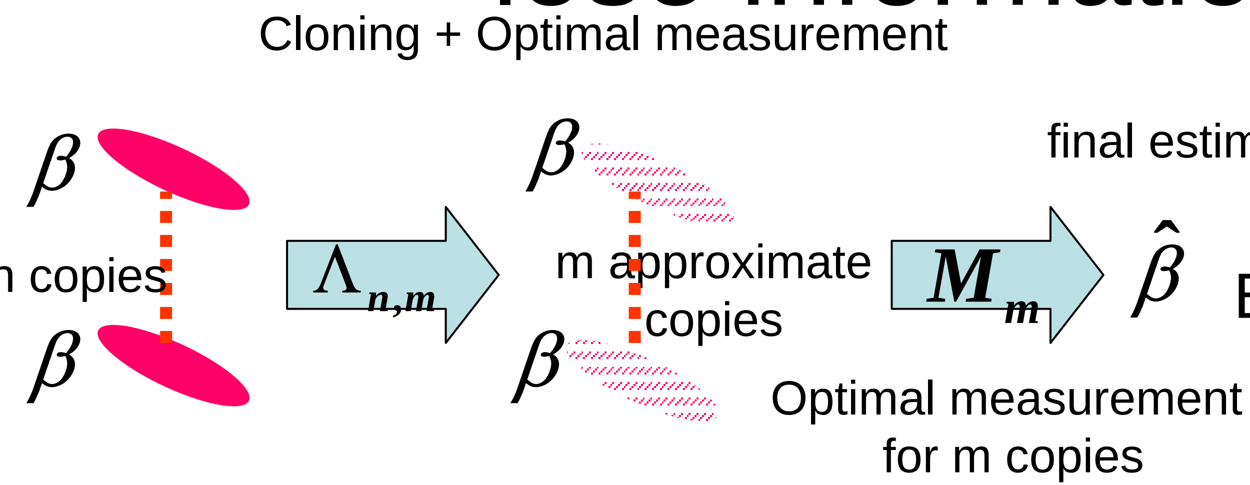
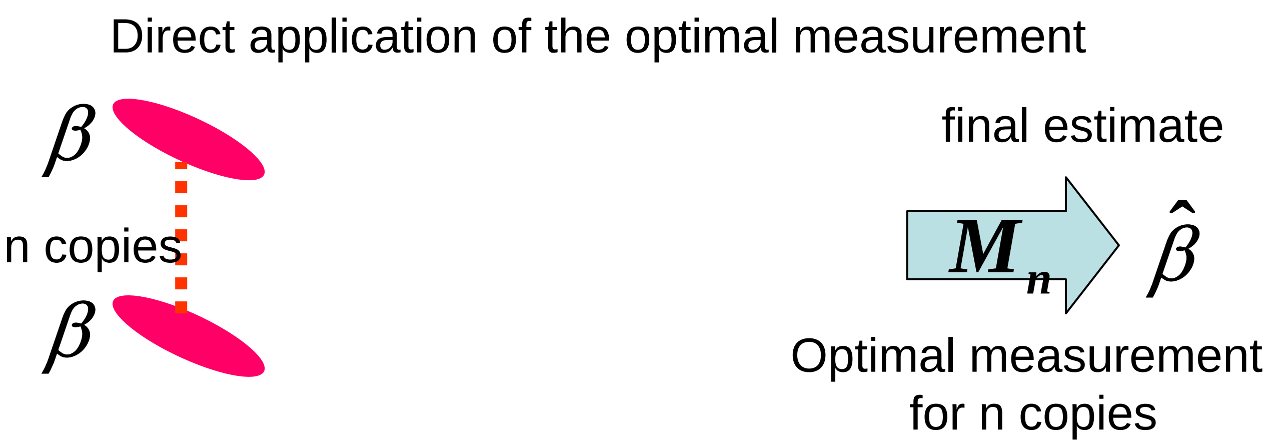
Optimal cloning of squeezed state



Optimal cloning does not lose information



Both distributions of $\hat{\beta}$ are coincident.



The same result can be shown in the qubit (qudit) case and the coherent state case.

Optimal cloning of squeezed state

Initial family $\left\{ \left(|\beta\rangle_{s\ s}\langle\beta| \right)^{\otimes n} \right\}$**, Target family** $\left\{ \left(|\beta\rangle_{s\ s}\langle\beta| \right)^{\otimes m} \right\}$.

Λ **: covariant, i.e.,**

$$\Lambda(V(g)^{\otimes n}\rho(V(g)^{\otimes n})^{\dagger}) = V(g)^{\otimes m}\Lambda(\rho)(V(g)^{\otimes m})^{\dagger}.$$

Optimize the fidelity ${}_s\langle\beta|^{\otimes m}\Lambda\left(\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\right)|\beta\rangle_s^{\otimes m}$.

If $m \geq n > 2,$ **the optimal cloning is**

$$\Lambda_{n,m}(\rho) \triangleq \frac{n-2}{m-2} P_m\left(\rho \otimes I^{\otimes(m-n)}\right) P_m.$$

$$\Lambda_{n,m}\left(\left(|0\rangle_{s\ s}\langle 0|\right)^{\otimes n}\right) = \sum_{k=0}^{\infty} \frac{(k+\frac{m-n}{2}-1)\cdots\frac{m-n}{2}(n-2)}{(k+\frac{m}{2}-1)\cdots\frac{m}{2}(m-2)} |k\rangle_{N\ N}\langle k|,$$

where $|k\rangle_m \triangleq \frac{1}{c_k}\left(\left(a^{(m)}\right)^{\dagger}\right)^k |0\rangle^{\otimes m}$.

$${}_s\langle\beta|^{\otimes m}\Lambda_{n,m}\left(\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\right)|\beta\rangle_s^{\otimes m} = \frac{n-2}{m-2}$$

Optimal cloning does not lose information

The optimal asymptotic error of the family

$$\left\{\Lambda_{n,m}\left(\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\right)\right\}$$ **equals that of family** $$\left\{\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\right\}.$$

That is, if we perform the measurement $\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes m}$ **for the family** $\left\{\Lambda_{n,m}\left(\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\right)\right\}$, **the data obey the distribution**

$$\left(\frac{n}{2}-1\right)\left(1-\left|\frac{\beta-\hat{\beta}}{\beta\overline{\hat{\beta}}-1}\right|^2\right)^{\frac{n}{2}}\frac{d^2\hat{\beta}}{\pi(1-|\hat{\beta}|^2)^2},$$

which gives the distribution when we apply the optimal measurement to *n* copies.
Similar phenomenon happens in the case of coherent state.

Its reason

We focus on the dual map $\Lambda_{n,m}{}^*$ of $\Lambda_{n,m}$.

$$(m-2)\Lambda_{n,m}{}^*\left(\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes m}\right)\frac{d^2\beta}{\pi(1-|\beta|^2)}$$

$$=(m-2)\frac{n-2}{m-2}\mathrm{Tr}_{H^{\otimes(m-n)}}\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes m}\frac{d^2\beta}{\pi(1-|\beta|^2)}$$

$$=(n-2)\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\frac{d^2\beta}{\pi(1-|\beta|^2)}$$

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