THE ASYMPTOTIC BEHAVIOUR OF MASTER EQUATIONS

(Quantum Markov Semigroups)

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WORKSHOP ON QUANTUM OPEN SYSTEMS

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1. Quantum Markov (Dynamical) Semigroups

2. Recurrent and transient projection

3. Irreducible components

4. Uniqueness of invariant states

5. Convergence to invariant states

6. Exponential ergodicity

7. Generic generator ...

8. 2-photon absorption and emission process

Quantum (Markov) semigroup

$\mathcal{A}$ von Neumann algebra $\quad (\mathcal{A} = (\mathcal{A}_*)^*)$

$$\mathcal{T}_t : \mathcal{A} \to \mathcal{A}, \quad t \geq 0$$

i) (semigroup)

$$\mathcal{T}_0(a) = a, \quad \mathcal{T}_t(\mathcal{T}_s(a)) = \mathcal{T}_{t+s}(a),$$

ii) (positive) $\quad a \geq 0 \quad \Rightarrow \quad \mathcal{T}_t(a) \geq 0,$

iii) (unital) $\quad \mathcal{T}_t(\mathbb{1}) = \mathbb{1}$

iv) (normal) $\quad a_\alpha \uparrow a \quad \Rightarrow \quad \mathcal{T}_t(a_\alpha) \uparrow \mathcal{T}_t(a),$

v) (w^* continuous) $\quad t \to \mathcal{T}_t(a) \quad \forall a$

$\mathcal{T}$ is *subMarkov* if $\mathcal{T}_t(\mathbb{1}) \leq \mathbb{1}$

Classical System.

$$\mathcal{A} = L^\infty(E, \mathcal{E}, \mu), \qquad \mathcal{A}_* = L^1(E, \mathcal{E}, \mu).$$

Quantum Open System (typical)

$$\mathcal{A} = \mathcal{B}(\mathsf{h}), \qquad \mathcal{A}_* = L^1(\mathsf{h}) \text{ (trace class)}.$$

Often positivity ii) is replaced by the stronger

sp) (strongly positive or Schwarz)

$$\mathcal{T}_t(a^*)\mathcal{T}_t(a) \le \mathcal{T}_t(a^*a) \quad \forall a \in \mathcal{A}$$

cp) (complete positivity)

$\forall t \ge 0$, $\forall n \in \mathbb{N}$, $a_{jk} \in \mathcal{A}$ $(1 \le j, k \le n)$

$$\begin{pmatrix} a_{11} & \cdot & a_{1n} \\ \cdots & \cdot & \cdots \\ a_{n1} & \cdot & a_{nn} \end{pmatrix} \ge 0, \Rightarrow \begin{pmatrix} \mathcal{T}_t(a_{11}) & \cdot & \mathcal{T}_t(a_{1n}) \\ \cdots & \cdot & \cdots \\ \mathcal{T}_t(a_{n1}) & \cdot & \mathcal{T}_t(a_{nn}) \end{pmatrix} \ge 0$$

$(E, \mathcal{E}, \mu)$ (μ σ-finite) measure space

$$\begin{pmatrix} \text{Markov process} \\ \text{values in } E \end{pmatrix} \approx \begin{pmatrix} \text{Markov semigroup} \\ \text{on } L^\infty(E, \mathcal{E}, \mu) \end{pmatrix}$$

$L^\infty(\cdots)$ commutative von Neumann algebra

$\mathcal{A}$ von Neumann algebra

$$\begin{pmatrix} \text{Q-Markov proc.} \\ \text{values in } \mathcal{A} \end{pmatrix} \approx \begin{pmatrix} \text{Q-M. semigroup} \\ \text{on } \mathcal{A} \end{pmatrix}$$

Problems:

- the Q-M. process goes to ∞ (fast/slow) ?
- or wanders in a “bounded” region ?
- what about invariant states ?
- convergence to an invariant state ?

Classical Markov chain (finite)

Blocks of the transition matrix T

$$\begin{array}{lcc} \text{transient states} & S & \to \\ \text{recurrent class} & R_1 & \to \\ \text{recurrent class} & R_2 & \to \end{array} \left(\begin{array}{c|c|c} * & * & * \\ \hline 0 & * & 0 \\ \hline 0 & 0 & * \end{array}\right)$$

μ initial distribution

$$\frac{1}{n}\sum_{k=1}^{n} T^{*k}\mu \underset{n \to \infty}{\longrightarrow} \nu$$

- $\text{supp}(\nu) \subseteq R_1 \cup R_2$,
- if $\text{supp}(\mu) \subseteq R_k \Rightarrow \text{supp}(\nu) \subseteq R_k \ \forall k = 1, 2$,
- $T1_{R_k} \geq 1_{R_k}$ $(\forall k = 1, 2)$ and $T1_S \leq 1_S$,
- $T^n 1_S \downarrow 0$ (for $n \to \infty$) and $\sum_n T^n 1_S < \infty$.

T $d \times d$ stochastic matrix acts on

$\mathcal{A} = \ell^\infty(\{1, \ldots, d\}; \mathbb{C}^d)$ as

$(Tf)(j) = \sum_k T_{jk} f(k)$.

T is called *irreducible* if

$\forall i, j \in \{1, \ldots, d\}$, $\exists\ n > 0$ s.t. $(T^n)_{ij} > 0$

Let $\mathcal{A}_I = \{\, f \in \mathcal{A} \mid f(k) = 0, \forall k \notin I\}$

The following are equivalent:

1. T *is irreducible,*

2. *there exists no* **proper** *subset* I *of* $\{1, \ldots, d\}$ *such that* $T(\mathcal{A}_I) \subseteq \mathcal{A}_I$.

$\mathcal{A}_I$'s are all the *hereditary* subalgebras of $\mathcal{A}$

i.e. $\mathcal{B} = p\mathcal{A}p$ for p projection in $\mathcal{A}$

Let $p = 1_I$. Then (a)$\Longleftrightarrow$(b)

(a) $T(\mathcal{A}_I) \subseteq \mathcal{A}_I$

(b) $Tp \leq p$

Proposition. Φ positive linear on $\mathcal{A}$ with $\Phi(\mathbb{1}) \leq \mathbb{1}$ and let $p \in \mathcal{A}$ be a projection. Then

$$\Phi(p\mathcal{A}p) \subseteq p\mathcal{A}p \quad \Longleftrightarrow \quad \Phi(p) \leq p.$$

A projection p s.t. $\Phi(p) \leq p$ is *superharmonic*

Definition. $\mathcal{T}$ is *irreducible* iff $\mathcal{T}_t(p) \leq p \ \forall t \geq 0$ implies either $p = \mathbb{1}$ or $p = 0$.

Goal.

Decompose QMS into irreducible "sub"-QMS.

Fast recurrent projection

$\mathcal{T}$ QMS on a vN algebra $\mathcal{A}$

The support projection p_ω of a normal invariant state ω is subharmonic $\mathcal{T}_t(p_\omega) \geq p_\omega$ $\forall t \geq 0$.

Prop. Let p_α be subharmonic projections. Then $p := \sup_\alpha p_\alpha$ is a subharmonic projection.

$\mathcal{S}$ normal invariant states

Defn. *Fast recurrent projection*

$$p_R := \sup_{\omega \in \mathcal{S}} p_\omega, \quad (p_\omega \text{ support of } \omega).$$

Subharmonicity $\mathcal{T}_t(p_R) \geq p_R \Rightarrow$ ($\mathcal{T}_*$ predual sgr)

$$\boxed{\text{supp}(\mathcal{T}_{*t}(\sigma)) \subseteq p_R}$$

for all state σ with supp(σ) $\subseteq p_R$.

Fast recurrent part of $\mathcal{T}$

QMS on $p_R \mathcal{A} p_R$ $\qquad$ $\mathcal{T}_t^R := \left(\mathcal{T}_{*t}\Big|_{p_R \mathcal{A}_* p_R} \right)^*$

$$\mathcal{T}_t^R(b) = p_R \mathcal{T}_t(b) p_R \quad \forall b = p_R a p_R,\ a \in \mathcal{A}$$

Prop. $\forall a \in \mathcal{A}$ there exists

$$w^* - \lim_{t \to \infty} \frac{1}{t} \int_0^t p_R \mathcal{T}_s(a) p_R ds$$

and defines a conditional expectation

$$\mathcal{E} : \mathcal{A} \to \mathcal{F}(\mathcal{T}^R).$$

($\mathcal{F}(\mathcal{T}^R)$ fixed points of $\mathcal{T}^R$).

Decomposition of $\mathcal{T}^R$

$$\begin{array}{lcc} \text{recurrent class } p_1 & \to \\ \text{recurrent class } p_2 & \to \\ \text{recurrent class } p_3 & \to \\ \text{recurrent class } \ldots & \to \end{array} \left(\begin{array}{c|c|c|c} * & 0 & 0 & \cdots \\ \hline 0 & * & 0 & \cdots \\ \hline 0 & 0 & * & \cdots \\ \hline \cdots & \cdots & \cdots & \cdots \end{array} \right)$$

Thm. Suppose $\mathcal{A} \subseteq \mathcal{B}(\mathsf{h})$ σ-finite and

either $\mathcal{F}(\mathcal{F}_*)$ is finite dimensional,

or $\mathcal{A}$ is abelian

Then there exists a sequence $(p_n)_{n\geq 1}$ of orthogonal projections in h such that

1.
$$p_R = \sum_{n\geq 1} p_n, \qquad \mathcal{T}_t^R(p_n) = p_n \quad \forall n,$$

2. the reduced semigroup
$$\mathcal{T}_t^{p_n}(x) := p_n \mathcal{T}_t(x) p_n$$
(on $p_n \mathcal{A} p_n$) is irreducible.

$\forall n$ $\mathcal{T}^{p_n}$ (QMS on $p_n \mathcal{A} p_n$) has a *unique* faithful normal invariant state.

Form potential ($\mathcal{A}$ acts on a Hilbert space h)

$$\mathcal{U}_f(x)[u] = \int_0^\infty \langle u, \mathcal{T}_t(x)u\rangle dt \quad x \in \mathcal{A}_+,\ u \in \mathsf{h}.$$

Prop. Projections onto $\overline{\{ u \in \mathsf{h} \mid \mathcal{U}_f(x)[u] < \infty\}}$ and $\{ u \in \mathsf{h} \mid \mathcal{U}_f(x)[u] = 0\}$ are in $\mathcal{A}$ and are *subharmonic*.

If $\mathcal{U}_f(x)$ is densely defined we find a *selfadjoint* operator $\mathcal{U}(x)$ representing the form $\mathcal{U}_f(x)$.

$$\mathcal{A}_{\mathrm{int}} := \left\{ x \in \mathcal{A}_+ \mid \mathcal{U}(x) \text{ bounded } \right\}$$

$$\mathcal{P}_{\mathrm{tr}} := \{p \text{ proj} \in \mathcal{A} \mid \exists x \in \mathcal{A}_{\mathrm{int}} \text{ s.t. } p = \mathrm{supp}(\mathcal{U}(x))\}$$

Defn. *Transient projection* associated with $\mathcal{T}$

$$\boxed{p_T := \sup_{p \in \mathcal{P}_{\mathrm{tr}}} p}$$

Remark. $p_T \leq p_R^{\perp}$.

Thm. If $\mathcal{A}$ is σ-finite then

1. there exists an increasing sequence $(p_n)_{n\geq 1}$, $p_n \in \mathcal{A}_{\mathsf{int}}$ $\forall n$ s.t. $p_T = \sup_n p_n$,

2. p_T is *superharmonic* i.e. $\mathcal{T}_t(p_T) \leq p_T$ $\forall t$.

Rem. Transient part. $\mathcal{T}_t^T := \mathcal{T}_t\Big|_{p_T \mathcal{A} p_T}$.

Defn. *slow recurrent projection*

$$p_{R_0} := p_R^{\perp} - p_T$$

Defn. A QMS $\mathcal{T}$ is called

transient	if	$p_T = \mathbb{1}$,
recurrent	if	$p_T = 0$,
slow recurrent	if	$p_{R_0} = \mathbb{1}$,
fast recurrent	if	$p_R = \mathbb{1}$.

Prop. An irreducible QMS $\mathcal{T}$ is either transient, slow recurrent or fast recurrent.

Potentials and superharmonic operators.

$$\begin{aligned}\mathcal{T}_t(\mathcal{U}(x)) &= \mathcal{T}_t\left(\int_0^\infty \mathcal{T}_s(x)ds)\right) \\ &= \int_t^\infty \mathcal{T}_s(x)ds \le \mathcal{U}(x)\end{aligned}$$

Proposition. *An irreducible QMS $\mathcal{T}$ is transient if and only if it admits a non-trivial superharmonic operator.*

Corollary. $\mathcal{A} = \mathcal{B}(\mathbf{h})$, h *separable. A transient* $\mathcal{T}$ *has no normal invariant state.*

Rem. $\exists\, \mathcal{U}(p) \Rightarrow \mathcal{T}_t(p) \to_{t\to\infty} 0$.

Indeed, $\mathcal{U}(p_n)$ bounded implies $\mathcal{T}_t(p_n) \to_n 0$. Thus, if $\omega(\mathcal{T}_t(a)) = \omega(a)\ \forall t \ge 0$, then $\exists \nu$ s.t. $\omega(p_n) > 1/2\ \forall n > \nu$ and

$$1/2 < \limsup_n \omega(p_n) = \limsup_n \omega(\mathcal{T}_t(p_n)) = 0.$$

$\mathcal{A}$ σ-finite ($\mathcal{T}$ could be 2-positive)

Thm. The following are equivalent:

T1 $\exists\ x \in \mathcal{A}_+$, with $\mathcal{U}(x)$ bounded and $\mathcal{U}(x) > 0$,

T2 $\exists$ a positive $x \in \mathcal{A}$ with $\mathcal{U}(x) > 0$,

T3 $\exists$ an increasing sequence of projections $(p_n)_{n\geq 1}$, with $p_n \uparrow \mathbb{1}$ and $\mathcal{U}(p_n)$ bounded $\forall n$.

Thm. The following are equivalent:

R1 $\forall\ x \in \mathcal{A}_+$ and $u \in \mathsf{h}$ either $\mathcal{U}_f(x)[u] = +\infty$ or $\mathcal{U}_f(x)[u] = 0$,

R2 $\forall$ projection $p \in \mathcal{A}$ and $u \in \mathsf{h}$ either $\mathcal{U}_f(p)[u] = +\infty$ or $\mathcal{U}_f(p)[u] = 0$,

Rem. If $\mathcal{A} = \mathcal{B}(\mathsf{h})$, with h separable, projections above can be taken finite dimensional.

Asymptotic behaviour

Defn. A QMS $\mathcal{T}$ is:

a) *mean ergodic* if $\exists$

$$w^* - \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{T}_s(a)ds \qquad \forall a \in \mathcal{A}.$$

b) *ergodic* if $\exists$

$$w^* - \lim_{t\to\infty} \mathcal{T}_t(a) \qquad \forall a \in \mathcal{A}.$$

Predual semigroup $\mathcal{S} = (\mathcal{S}_t)_{t\geq 0}$: the unique sgr on $\mathcal{A}_*$ s.t. $(\mathcal{S}_t)^* = \mathcal{T}_t$.

Prop. Any w-limit ρ of the Césaro means

$$\frac{1}{t_\alpha}\int_0^{t_\alpha} \mathcal{S}_s(\sigma)ds \quad (\sigma \in \mathcal{A}_*)$$

is $\mathcal{S}$-invariant i.e. $\mathcal{S}_t(\rho) = \rho\ \forall t$.

• *Support projection* of $\omega \in \mathcal{A}_*$: smallest proj. $p \in \mathcal{A}$ s.t. $\omega(a) = \omega(pa) = \omega(ap) = \omega(pap)\ \forall a$.

$\mathcal{T}$ Schwarz QMS - Q1: $\exists$ invariant state ?

Rec. An invariant state exists iff $\exists$ a non-zero positive $\omega \in \mathcal{A}_*$ s.t. $(t^{-1}\int_0^t \mathcal{S}_s(\omega)ds; t \geq 0)$ is w-relatively compact.

Thm. (FF, Rebolledo 2001) $\mathcal{A} = \mathcal{B}(\mathsf{h})$ (h sep.). If there exists X, Y self-adjoint with $X \geq 0$, Y bounded form below with finite-dimensional spectral projections $E_Y]-\infty, r]$ s.t. $\forall\ t, r \geq 0$

$$\int_0^t \langle u, \mathcal{T}_s(Y \wedge r)u\rangle ds \leq \langle u, Xu\rangle$$

$\forall u \in \mathrm{Dom}(X)$ then $\mathcal{T}$ has an invariant state.

(Formally eq. to $\mathcal{L}(X) \leq -Y$; $\mathcal{L}$ generator).

Thm. (Emel'yanov, Wolff 2002) Suppose $\mathcal{A}$ semifinite. If there exists a $\eta \in \mathcal{A}_*$ s.t.

$$\lim_{t\to\infty} \left\| \left(\eta - t^{-1}\int_0^t \mathcal{S}_s(\omega)ds\right)_+ \right\| = 0$$

$\forall$ positive ω with $\|\omega\| = 1$ then $\mathcal{T}$ has an invariant state.

$\mathcal{T}$ Schwarz QMS, ρ faithful invariant state

uniqueness

$$\mathcal{F}(\mathcal{T}) := \{\, a \in \mathcal{A} \mid \mathcal{T}_t(a) = a \,\}$$

Prop.

1. $\mathcal{F}(\mathcal{T})$ is a vN subalgebra of $\mathcal{A}$,

2. $\exists\ \mathcal{E} : \mathcal{A} \to \mathcal{F}(\mathcal{T})$ (Umegaki) cond. expect.

Thm. The following are equivalent

1. ρ is the unique normal $\mathcal{T}$-invariant state,

2. $\mathcal{F}(\mathcal{T}) = \mathbb{C} \cdot \mathbb{1}$,

3. $\lim_{t\to\infty} \frac{1}{t} \int_0^t \sigma(\mathcal{T}_s(a))ds = \sigma(\mathbb{1})\rho(a)$

   $\forall a \in \mathcal{A}, \sigma \in \mathcal{A}_*$.

$\mathcal{T}$ Schwarz QMS, ρ faithful invariant state

Q2: is $\mathcal{T}$ ergodic ?

$$\mathcal{N}(\mathcal{T}) := \{\, a \in \mathcal{A} \mid \mathcal{T}_t(a^*)\mathcal{T}_t(a) = \mathcal{T}_t(a^*a), \\ \mathcal{T}_t(a)\mathcal{T}_t(a^*) = \mathcal{T}_t(aa^*) \ \forall t \geq 0\}$$

Ex. $\mathcal{T}_t(a) = \mathrm{e}^{-itH}a\mathrm{e}^{itH}$ (H s.a.) $\Rightarrow \mathcal{N}(\mathcal{T}) = \mathcal{A}$

Thm. (Frigerio, Verri 1982) If $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$ then $\forall a \in \mathcal{A}$ there exists

$$w^* - \lim_{t\to\infty} \mathcal{T}_t(a)$$

Remark

- This suff. condition is often necessary.
- $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T}) \approx$ periph. spectr. of $\mathcal{T}_t = \{1\}$.

Next step: conditions on the generator of $\mathcal{T}$

- $\mathcal{T}_t(a) \le a$ iff $\mathcal{L}(a) \le 0$

- $\mathcal{T}_t(a^*)\mathcal{T}_t(a) = \mathcal{T}_t(a^*a)$ iff $a^*\mathcal{L}(a) + \mathcal{L}(a^*)a = \mathcal{L}(a^*a)$.

$$\pounds(x) = G^*x + \sum_{\ell=1}^{\infty} L_\ell^* x L_\ell + xG$$

- conditions on $G, L_1, \ldots$ for p subharmonic,

- relationship between $\mathcal{F}(\mathcal{T})$ and $G, L_1, \ldots$,

- relationship between $\mathcal{N}(\mathcal{T})$ and $G, L_1, \ldots$,

Lindblad's theorem

$\mathcal{A} = \mathcal{B}(\mathsf{h})$ (h sep.) and $\mathcal{T}$ completely positive norm-continuous i.e.

$$\lim_{t\to 0} \sup_{x\in\mathcal{B}(\mathsf{h}),\|x\|\le 1} \|\mathcal{T}_t(x) - x\| = 0$$

semigroup with *infinitesimal generator*

$$\mathcal{L} : \mathcal{B}(\mathsf{h}) \to \mathcal{B}(\mathsf{h}), \quad \mathcal{L}(x) = \text{norm} - \lim_{t\to 0} \frac{\mathcal{T}_t(x) - x}{t}.$$

$$\Rightarrow \quad \mathcal{L}(x) = i[H, x] + \frac{1}{2}\sum_{\ell=1}^{\infty} \left(-L_\ell^* L_\ell x + 2L_\ell^* x L_\ell - x L_\ell^* L_\ell\right)$$

where $i[H,x] = i(Hx - xH)$ and

- L_ℓ, $\ell = 1, 2, \ldots$, and H are in $\mathcal{B}(\mathsf{h})$,
- $H = H^*$ and $\sum_\ell L_\ell^* L_\ell$ converges strongly on h.

Conversely such an $\mathcal{L}$ generates a norm-cont. $\mathcal{T}$

Remark.

$\mathcal{T}_t(\mathbb{1}) \le \mathbb{1}$ (resp. $= \mathbb{1}$) iff $\mathcal{L}(\mathbb{1}) \le 0$ (resp. $= 0$).

Writing $\quad G = -\frac{1}{2}\sum_{\ell\geq 1} L_\ell^* L_\ell - iH \quad$ then

$$\mathcal{L}(x) = G^* x + \sum_{\ell\geq 1} L_\ell^* x L_\ell + xG.$$

Subharmonic projections

$\mathcal{T}_t(p) \geq p$ iff $\mathcal{L}(p) \geq 0$ iff $\mathcal{L}(p^\perp) \leq 0$ then

$$\begin{aligned} 0 \geq p\mathcal{L}(p^\perp)p &= p\,G^* p^\perp p + \sum_{\ell\geq 1} pL_\ell^* p^\perp L_\ell p + pp^\perp Gp \\ &= \sum_{\ell\geq 1} pL_\ell^* p^\perp L_\ell p \end{aligned}$$

then $p^\perp L_\ell p = 0$ (p is L_ℓ-invariant $\forall \ell$). Moreover $\mathcal{L}(p^\perp) \leq 0$ and $p\mathcal{L}(p^\perp)p = 0$ implies

$$\mathcal{L}(p^\perp)p = 0 \qquad \Rightarrow \qquad p^\perp Gp = 0.$$

Prop. *A projection p is subharmonic iff p is an invariant subspace for G and all the $L_\ell' s$.*

If p is a *maximal* subharmonic projection reduce $\mathcal{T}$ to $p^\perp \mathcal{A} p^\perp$ and go on ...

Existence of invariant states

$$\int_0^t \mathcal{T}_s(Y)ds \le X \quad \Leftrightarrow \quad \mathcal{L}(X) \le -Y.$$

$\Rightarrow$ Differentate at $t = 0$

$$\int_0^t \mathcal{T}_s(Y)ds \le X - \mathcal{T}_t(X).$$

$$\Leftarrow \qquad \frac{d}{dt}\left(\mathcal{T}_t(X) - X + \int_0^t \mathcal{T}_s(Y)ds\right) = \mathcal{T}_t\left(\mathcal{L}(X) + Y\right) \le 0.$$

Thm. (FF, Rebolledo JMP 2001) $\mathcal{A} = \mathcal{B}(\mathsf{h})$ (h sep.). If there exists X, Y self-adjoint with $X \ge 0$, Y bounded form below with finite-dimensional spectral projections $E_Y]-\infty, r]$ s.t. $\mathcal{L}(X) \le -Y$ then $\mathcal{T}$ has an invariant state.

(add. assumptions for unbd. Lindblad type $\mathcal{L}$)

Rem. If $\mathcal{T}$ is irreducible any normal $\mathcal{T}$-invariant state ρ is *faithful*. (The support projection of ρ is subharmonic).

$\mathcal{T}$ norm-continuous QMS, $\mathcal{L}$ Lindblad generator

ρ faithful normal $\mathcal{T}$-invariant state

Thm. $\mathcal{F}(\mathcal{T}) = \{H, L_1, L_1^*, L_2, L_2^*, \ldots\}'$ *and*
$\mathcal{N}(\mathcal{T}) = \{L_1, L_1^*, L_2, L_2^*, \ldots\}'$

Thm. (FF, Rebolledo IDAQP '98) (*). *If*

$$\{H, L_1, L_1^*, L_2, L_2^*, \ldots\}' = \{L_1, L_1^*, L_2, L_2^*, \ldots\}'$$

then $\mathcal{T}$ is ergodic ($\mathcal{T}_t(a)$ w^*-conv. for $t \to \infty$). If, moreover, $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$ then ρ is the unique faithful normal invariant state.

(*) Additional assumptions also for non norm-continuous $\mathcal{T}$

Example: decay to vacuum

$\mathsf{h} = \ell^2(\mathbb{N})$ c.o.n. basis $(e_n)_{n\geq 0}$. Generator

$$\mathcal{L}(x) = i[H, x]$$
$$-\frac{1}{2}\sum_{l<m} \gamma_{lm} \left(\{|e_m\rangle\langle e_m|, x\} - 2|e_m\rangle\langle e_l|x|e_l\rangle\langle e_m|\right)$$

where, $H := \sum_{m\geq 0} \varepsilon_m |e_m\rangle\langle e_m|, \varepsilon_m \in \mathbb{R}$, $\gamma_{lm} \geq 0$.

Origin: stochastic limit of
(reservoir) + (system) + (dypole interaction)

Define the number operator $N := \sum_m m|e_m\rangle\langle e_m|$.
For "good" functions $f \in \ell^\infty(\mathbb{N};\mathbb{C})$

$$\begin{aligned}\mathcal{L}(f(N)) &= \sum_m \sum_{l<m} \gamma_{lm}(f(l) - f(m))|e_m\rangle\langle e_m| \\ &= \sum_{k>0} \gamma_{N-k,N}(f(N-k) - f(N))\end{aligned}$$

$$\longleftarrow \boxed{\text{m-1}} \longleftarrow \boxed{\text{m}} \longleftarrow \boxed{\text{m+1}} \longleftarrow \boxed{\text{m+2}} \longleftarrow$$

Example: decay to vacuum 2

L_ℓ's here are the $L_{l\,m} = \gamma_{l\,m}^{1/2} |e_l\rangle\langle e_m|$ and

$$G = -\frac{1}{2}\left(\sum_m \left(\sum_{l<m} \gamma_{lm}\right)|e_m\rangle\langle e_m|\right) - iH$$

Note that $[G, N] = 0$ (N number operator).

p Subharmonic projection

$\Rightarrow \gamma_{lm}^{1/2} |p^\perp e_l\rangle\langle p e_m| = 0 \quad \forall\, l < m$

$\Leftrightarrow$ if $\gamma_{lm} > 0$ either $p e_m = 0$ or $p^\perp e_l = 0$

$\Leftrightarrow$ if $\gamma_{lm} > 0$ either $p^\perp e_m = e_m$ or $p e_l = e_l$

Suppose, e.g., $\gamma_{l\,l+1} > 0\ \forall l$ then $p^\perp e_{l+1} = e_{l+1}$ or $p e_l = e_l$ i.e. p has the form

$$p = \sum_{j=0}^{k} |e_j\rangle\langle e_j|, \qquad k \in \mathbb{N} \cup \{+\infty\}$$

unbounded L_ℓ, H

Hypothesis H-min

(i) G generates a strongly cont. sgr (P_t) on h,

(ii) $\mathrm{Dom}(L_\ell) \supseteq \mathrm{Dom}(G)$ $\forall \ell \geq 1$,

(iii) $\exists$ $c \in \mathrm{R}$ constant s.t. $\forall u \in \mathrm{Dom}(G)$

$$2\Re e\langle u, Gu\rangle + \sum_{\ell \geq 1} \|L_\ell u\|^2 \leq c\|u\|^2.$$

Define

$$\pounds(x)[v,u] := \langle v, xGu\rangle + \sum_{\ell} \langle L_\ell v, xL_\ell u\rangle + \langle Gv, xu\rangle$$

Thm Under **Hyp. A** $\exists$ a QDS satisfying

$$\langle v, \mathcal{T}_t(x)u\rangle = \langle v, xu\rangle + \int_0^t \pounds(\mathcal{T}_s(x))[v,u]\, ds$$

$\forall u, v \in \mathrm{Dom}(G)$, $\forall x \in \mathcal{B}(\mathsf{h})$.

The QDS satisfying

$$\langle v, \mathcal{T}_t(x)u\rangle = \langle v, xu\rangle + \int_0^t \pounds(\mathcal{T}_s(x))[v,u]\, ds \qquad (E)$$

a) not nec. unique, b) might satisfy $\mathcal{T}_t(\mathbb{1}) < \mathbb{1}$.

Prop. *The following are equivalent:*

1. $\mathcal{T}$ *is the unique solution to (E),*
2. $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$,
3. *the domain of the generator* $\mathcal{L}$ *of* $\mathcal{T}$ *is*

$$\{\, x \in \mathcal{B}(\mathsf{h}) \mid \pounds(x) \text{ is bounded } \}.$$

Thm. (FF, AMC) (Techical assumptions ...)
Suppose that $\exists$ *a self-adjoint* X *such that:*

$$\sum_\ell L_\ell^* L_\ell \le X, \quad \pounds(X) \le bX.$$

(b *constant). Then* $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$.

$\mathsf{h} = \ell^2(I\!N)$, a annihilation, $a^\dagger$ creation, N number

$$\pounds(x) = G^* x + \sum_{\ell=1}^{2} L_\ell^* x L_\ell + xG$$

$$\begin{aligned} D(G) &= D(N^2), \\ G &= -\frac{\lambda^2}{2} a^2 a^{+2} - \frac{\mu^2}{2} a^{+2} a^2 - i\omega a^{+2} a^2 \end{aligned}$$

$\lambda \geq 0, \mu > 0$, $\omega \in I\!R$.

$$D(L_1) = D(L_2), \quad L_1 = \mu a^2, \quad L_2 = \lambda a^{+2}.$$

Regular if $\lambda \leq \mu$ ... $X = N^2$, $\pounds(X) \leq bX$.

Not regular if $\lambda > \mu$

Subharmonic projections if $\lambda > 0$

$$0,\ p_e = \sum_{k \geq 0} |e_{2k}\rangle\langle e_{2k}|,\ p_o = \sum_{k \geq 0} |e_{2k+1}\rangle\langle e_{2k+1}|,\ \mathbb{1}$$

Indeed $\mathcal{T}_t(p_e) = p_e$, $\mathcal{T}_t(p_o) = p_o$.

$$\mathcal{A}_e := p_e \mathcal{A} p_e, \qquad \mathcal{A}_o := p_o \mathcal{A} p_o,$$

$$\mathcal{T}_t(\mathcal{A}_e) \subseteq \mathcal{A}_e, \qquad \mathcal{T}_t(\mathcal{A}_o) \subseteq \mathcal{A}_o.$$

$$\mathcal{T}_t^e = \mathcal{T}_t\Big|_{\mathcal{A}_e}, \qquad \mathcal{T}_t^o = \mathcal{T}_t\Big|_{\mathcal{A}_o}$$

Faithful invariant states ($\nu = \lambda/\mu < 1$)

$$\mathcal{T}_t^e \qquad \rho_e = (1-\nu^2)\sum_{k=0}^{\infty} \nu^{2k} |e_{2k}\rangle\langle e_{2k}|$$

$$\mathcal{T}_t^o \quad \rho_o = (1-\nu^2)\sum_{k=0}^{\infty} \nu^{2k} |e_{2k+1}\rangle\langle e_{2k+1}|$$

$$\mathcal{F}(\mathcal{T}^e) = \mathcal{N}(\mathcal{T}^e) = \mathbb{C}\cdot \mathbb{1} = \mathcal{F}(\mathcal{T}^o) = \mathcal{F}(\mathcal{T}^o)$$

Then ρ_e (resp. ρ_o) is the unique invariant state of $\mathcal{T}^e$ (resp. $\mathcal{T}^o$) and, for any even state σ_e on $\mathcal{A}_e$ (resp. odd state σ_o on $\mathcal{A}_o$), we have

$$\lim_{t\to\infty} \mathcal{T}_{*t}^e(\sigma_e) = \rho_e, \qquad \lim_{t\to\infty} \mathcal{T}_{*t}^o(\sigma_o) = \rho_o.$$

$\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T}) = \mathrm{span}\{\, p_e, p_o \,\}$.

Thm. For $\lambda > 0$ all the $\mathcal{T}$-invariant states are

$$\alpha\rho_e + (1-\alpha)\rho_o, \qquad \alpha \in [0,1]$$

and, for all state σ on $\mathcal{A}$,

$$w - \lim_{t\to\infty} \mathcal{T}_{*t}(\sigma) = \mathrm{tr}(\rho_e p_e)\rho_e + \mathrm{tr}(\rho_o p_o)\rho_o.$$

If $\lambda = 0$ *all* subharmonic proj. are generated by

$$\sum_{k=0}^{n} |e_{2k}\rangle\langle e_{2k}|, \qquad \sum_{k=0}^{m} |e_{2k+1}\rangle\langle e_{2k+1}|,$$

$(n, m \in I\!N \cup \{+\infty\})$. *All* the invariant states are

$$\sum_{0\le j,k\le 2} \sigma_{jk}|e_j\rangle\langle e_k|,$$

$\sigma_{00}, \sigma_{11} \ge 0, \sigma_{00} + \sigma_{11} = 1, \sigma_{01} = \bar{\sigma}_{10}, |\sigma_{01}|^2 \le \sigma_{00}\sigma_{11}$.

Moreover

$$w - \lim_{t\to\infty} \mathcal{T}_{*t}(\eta)$$

exist ... exponential speed.

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