QUANTUM MARKOV SEMIGROUPS: STRUCTURE AND ASYMPTOTICS

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Abstract. We study the structure of a quantum Markov semigroup $(\mathcal{T}_t)_{t\geq 0}$ on a von Neumann algebra $\mathcal{A}$ starting from its decomposition by means of the transient and recurrent projections. The existence of invariant states and convergence to invariant state is also discussed. Applications to quantum Markov semigroups with Lindblad type infinitesimal generator are analysed.

1. Introduction

Quantum Markov Semigroups (QMS) form a distinguished class of semigroups of operators acting on an operator algebra with a special positivity property. They arose in the theory of open quantum systems as a model for irreversible evolutions in quantum mechanics. In the extensive physical literature on the subject (see [3], [4], [5], [6], [11], [22], [23], [27] and the references therein) they are usually called quantum dynamical semigroups.

From a mathematical point of view, QMS are a natural generalisation of classical Markov semigroups on a function space, which is replaced, in quantum theory, by a (non-commutative) operator algebra. This generalisation gives a rigorous basis to the study of the qualitative behaviour of evolution equations (master equations) on an operator algebra that presently, in the physical literature, are either computed explicitly, whenever this is possible, or simulated numerically (see [26] and the references therein).

In this paper we shall illustrate some recent results on QMS.

We start giving the definition of a QMS on an arbitrary von Neumann $\mathcal{A}$ algebra, a strongly closed subalgebra of the algebra $\mathcal{B}(\mathsf{h})$ of all bounded operators on a separable Hilbert space h. The identity operator on $\mathcal{A}$ will be denoted by $\mathbb{1}$. Recall that any von Neumann algebra $\mathcal{A}$ is the dual of a Banach space usually denoted $\mathcal{A}_*$ (called the predual space) and the weak* topology on $\mathcal{A}$ is obviously defined.

Definition 1.1. *A* quantum dynamical semigroup (QDS) *on a von Neumann algebra $\mathcal{A}$ is a family $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of bounded operators on $\mathcal{A}$ with the following properties:*

1. $\mathcal{T}_0(a) = a$, *for all* $a \in \mathcal{A}$,
2. $\mathcal{T}_{t+s}(a) = \mathcal{T}_t(\mathcal{T}_s(a))$, *for all* $s, t \geq 0$ *and all* $a \in \mathcal{A}$,
3. $\mathcal{T}_t$ *is completely positive for all* $t \geq 0$,
4. $\mathcal{T}_t$ *is a normal operator on* $\mathcal{A}$ *for all* $t \geq 0$, *i.e. for every increasing net* $(a_\alpha)_\alpha$ *in* $\mathcal{A}$ *with* $\text{l.u.b.}a_\alpha = a \in \mathcal{A}$ *we have* $\text{l.u.b.}_\alpha \mathcal{T}_t(a_\alpha) = \mathcal{T}_t(a)$,
5. *for each* $a \in \mathcal{A}$, *the map* $t \to \mathcal{T}_t(a)$ *is continuous with respect to the weak* topology on* $\mathcal{A}$.

A quantum dynamical semigroup is Markov *if it is identity-preserving.*

A map $\Phi : \mathcal{A} \to \mathcal{A}$ is called completely positive if, for every $n \geq 1$, the map on the algebra $\mathcal{A} \otimes M_n$ of $\mathcal{A}$-valued $n \times n$ matrices

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \longrightarrow \begin{pmatrix} \Phi(a_{11}) & \dots & \Phi(a_{1n}) \\ \dots & \dots & \dots \\ \Phi(a_{n1}) & \dots & \Phi(a_{nn}) \end{pmatrix}$$

is positive, i.e. maps positive operators to positive operators. It is known that a positive map on a commutative von Neumann algebra is completely positive ([28]).

Note that, for a QMS $\mathcal{T}$, the operators $\mathcal{T}_t$ turn out to be contractions for the norm of $\mathcal{A}$ as in the classical commutative case.

When $\mathcal{A} = \mathcal{B}(\mathsf{h})$, then $\mathcal{A}$ is the dual space of the Banach space $\mathcal{A}_*$ of trace class operators on h. In this case the simplest example of a QMS is given by

$$\mathcal{T}_t(a) = \mathrm{e}^{itH} a \mathrm{e}^{-itH}$$

where H is a self-adjoint operator on h. It is easy to see that the above semigroup is weakly* continuous and, if the Hilbert space h is infinite dimensional, then it is strongly continuous if and only if H is bounded (in this case the above QMS is also uniformly (or norm) continuous). This is the main reason for assuming weak* continuity in (5).

It is worth noticing here that, due to the property (4), the QMS $\mathcal{T}$ is the dual semigroup of a strongly continuous semigroup on $\mathcal{A}_*$, denoted $\mathcal{T}_*$, therefore one could study the properties of the latter semigroup and state them for $\mathcal{T}$ by duality. However, we shall study $\mathcal{T}$ directly in order to stress the role of complete positivity.

By the duality $(\mathcal{A}_*, \mathcal{A})$, we can easily find an operator $\mathcal{L}_*$ on $\mathcal{A}_*$ with adjoint equal to $\mathcal{L}$. This is clearly the generator of the preadjoint semigroup $\mathcal{T}_*$. The equation

$$\frac{d}{dt}\mathcal{T}_{*t}(\rho) = \mathcal{T}_{*t}(\mathcal{L}_*(\rho))$$

is called in the physical literature *Markovian Master Equation*.

A *state* is a positive linear functional ω on $\mathcal{A}$ normalized by $\omega(\mathbb{1}) = 1$. It is called *normal* if it is weak*-continuous. It is *faithful* if $\omega(a) = 0$ for a positive $a \in \mathcal{A}$ implies $a = 0$. A normal state ω admits a *density*, a positive operator D_ω on h such that $\omega(a) = \mathrm{tr}\,(D_\omega a)$ for all $a \in \mathcal{A}$.

Definition 1.2. *A normal state ρ is* invariant *or* stationary *for a QMS $\mathcal{T}$ if* $\mathrm{tr}\,(\rho\mathcal{T}_t(a)) = \mathrm{tr}\,(\rho a)$ *for all $a \in \mathcal{A}$ and $t \geq 0$ (i.e. $\mathcal{T}_{*t}(\rho) = \rho$ for all $t \geq 0$).*

The following fundamental result due to G. Lindblad [25] characterises the generator of a uniformly continuous QMS on the von Neumann algebra $\mathcal{B}(\mathsf{h})$.

Theorem 1.1. *Let $\mathcal{T}$ be a uniformly continuous semigroup of normal operators on $\mathcal{B}(\mathsf{h})$. The following are equivalent:*

(1) *$\mathcal{T}$ is a QMS, i.e. the maps $\mathcal{T}_t$ ($t \geq 0$) are completely positive,*
(2) *the infinitesimal generator $\mathcal{L}$ can be represented in the form*

$$\mathcal{L}(x) = G^*x + \sum_{\ell\geq 1} L_\ell^* x L_\ell + xG \tag{1}$$

with $L_\ell, G \in \mathcal{B}(\mathsf{h})$, the series $\sum_\ell L_\ell^* L_\ell$ strongly convergent (i.e. $\sum_{\ell\geq 1} \|L_\ell v\|^2 < \infty$ for all $v \in \mathsf{h}$) and $G + G^* + \sum_\ell L_\ell^* L_\ell = 0$.

A generalisation of Lindblad's theorem (1.1) to weak* continuous semigroups is not known but uniform continuity is too restrictive in several physical applications. There are, however, constructions of a QMS with an unbounded generator $\mathcal{L}$ associated with unbounded operators G, L_ℓ on h (see Section 6). A special attention will be given to QMS on $\mathcal{B}(\mathsf{h})$ whose generators can be written in a generalised Lindblad form (1). This class is sufficiently wide to cover the applications ([1], [3], [5], [4], [6], [11], [22], [23], [27]) to Quantum Optics.

The aim of this paper is to present several recent results describing the structure of a QMS and the qualitative behaviour of the associated dynamics (existence of invariant states, convergence to invariant state, escape to infinity, ...).

The paper is organised as follows. In Section 2 we introduce the *positive recurrent projection* determined by the supports of normal invariant states and in Section 3 we define the *transient projection* determined, on the contrary, by states with finite life time. These projections are orthogonal and determine in a natural way two reduced sub-semigroups of the given QMS with quite different behaviours leading us to a notion of irreducibility.

In Section 4 we discuss some tools for proving the existence of normal invariant states and the ergodicity of the QMS (Section 5).

In Section 6 we describe the generalised Lindblad type generators associated with unbounded G and L_ℓ's. Then we give in Section 7 sufficient conditions for the existence of normal invariant states and the ergodicity of the QMS depending only on the operators G and L_ℓ which are usually given in the applications instead of the QMS.

A lot of information on the QMS $\mathcal{T}$ can be obtained by a simple analysis especially when the generator is associated with operators G, L_ℓ (see e.g. Theorem 6.1, Theorem 7.1, Theorem 7.2). Deeper spectral properties have been investigated in [7] and [9].

Some applications, drawn from the bewildering variety of Markovian Master Equations in the physical literature, have been studied in detail but will not be discussed here for lack of space. We refer the interested reader to [15], [16], [17], [19] .

2. The recurrent projection

Let ω be a normal state on $\mathcal{A}$. Its support projection $s(\omega)$ is defined as the smallest projection p in $\mathcal{A}$ such that $\omega(p) = 1$ (see [13]).

Proposition 2.1. *Let ρ be a normal invariant state for a QMS $\mathcal{T}$. The support projection $s(\rho)$ satisfies $\mathcal{T}_t(s(\rho)) \geq s(\rho)$ for all $t \geq 0$.*

Proof. Denote $p = s(\rho)$. We have then

$$\mathrm{tr}\,(\rho p \mathcal{T}_t(p^\perp)p) = \mathrm{tr}\,(\rho \mathcal{T}_t(p^\perp)) = \mathrm{tr}\,(\rho p^\perp) = 0.$$

It follows then that $p\mathcal{T}_t(p^\perp)p = 0$ because ρ is faithful on the algebra $p\mathcal{A}p$. The conclusion follows then from the simple Lemma 2.1. □

Lemma 2.1. *Let a be a positive element of $\mathcal{A}$ and let p be a projection in $\mathcal{A}$. If $pap = 0$ then $a = p^\perp a p^\perp$.*

Proof. Indeed, for all $v, u \in \mathsf{h}$ with $pu = u$ and $p^\perp v = v$, and all $z \in \mathbb{C}$ we have

$$0 \leq \langle (v + zu), a(v + zu)\rangle = 2\Re e z\langle v, au\rangle + \langle v, av\rangle.$$

This implies $\langle v, au\rangle = 0$. □

Following the terminologies of potential theory and Markov processes we define

Definition 2.1. *A self-adjoint element a of $\mathcal{A}$ is called* subharmonic *(resp.* superharmonic*,* harmonic*) for a QMS $\mathcal{T}$ if $\mathcal{T}_t(a) \geq a$ (resp. $\mathcal{T}_t(a) \leq a$, $\mathcal{T}_t(a) = a$) for all $t \geq 0$.*

The support projection of an invariant state is subharmonic (Proposition 2.1).

Definition 2.2. *Let* $(p_i)_{i\in I}$ *be a family of projections in a Hilbert space* h*. We denote by* $\vee_{i\in I} p_i$ *the projection onto the closure of the linear subspace of* h *generated by the ranges of the* p_i*'s.*

Proposition 2.2. *Let* $(p_i)_{i\in I}$ *be a family of subharmonic projections for a QMS* $\mathcal{T}$*. The projection* $p = \vee_{i\in I} p_i$ *is also subharmonic for* $\mathcal{T}$*.*

Proof. It suffices to check that $\mathcal{T}_t(p^\perp) \le p^\perp$. For all $u \in \mathsf{h}$ with $p_i u_i = u_i$, we have

$$\left\langle u_i, \mathcal{T}_t(p^\perp) u_i \right\rangle \le \left\langle u_i, \mathcal{T}_t(p_i^\perp) u_i \right\rangle \le \left\langle u_i, p_i^\perp u_i \right\rangle = 0.$$

Therefore, for all $v \in \mathsf{h}$ and all $u_i \in \mathsf{h}$ such that $p_i u_i = u_i$, we have

$$\left|\left\langle v, \mathcal{T}_t(p^\perp) u_i \right\rangle\right|^2 \le \|(\mathcal{T}_t(p^\perp))^{1/2} v\|^2 \cdot \|(\mathcal{T}_t(p^\perp))^{1/2} u_i\|^2 = 0.$$

It follows that $\mathcal{T}_t(p^\perp)p = 0$ and adjoint $p\mathcal{T}_t(p^\perp)p = 0$. The conclusion follows from Lemma 2.1. □

Definition 2.3. *We call* positive recurrent projection *associated with a QMS* $\mathcal{T}$ *the projection* $p_R = \vee_{i\in I} p_i$ *where the* p_i*'s are the support projections of all the invariant states of* $\mathcal{T}$*.*

If p is a $\mathcal{T}$-subharmonic projection and ω is a normal state with $s(\omega) \le p$. Then $s(\mathcal{T}_{*t}(\omega)) \le p$ for all $t \ge 0$. Indeed

$$\mathcal{T}_{*t}(\omega)(p^\perp) = \omega(\mathcal{T}_t(p^\perp)) \le \omega(p^\perp) = 0.$$

It follows that the restriction of the operators $\mathcal{T}_{*t}$ to $p\mathcal{A}_* p$ yields a trace preserving semigroup on $p\mathcal{A}_* p$. The dual semigroup $\mathcal{T}^p$ is a QMS on $p\mathcal{A}p$ characterized by $\mathcal{T}^p_t(a) = p\mathcal{T}_t(a)p$ for all $t \ge 0$. This will be called the *reduced semigroup* associated with the subharmonic projection p.

The reduced semigroup $\mathcal{T}^{p_R}$ associated with the positive recurrent projection p_R has a faithful set of normal invariant states. Moreover it is a Markov semigroup. Indeed

$$p_R = p_R \mathcal{T}_t(\mathbb{1}) p_R \ge p_R \mathcal{T}_t(p_R) p_R \ge p_R$$

Denote $\mathcal{F}(\mathcal{T})$ the linear space of fixed points of the QMS $\mathcal{T}$)

$$\mathcal{F}(\mathcal{T}) = \{ a \in \mathcal{A} \mid \mathcal{T}_t(a) = a, \ \forall t \ge 0 \}.$$

It is not hard to see that, if the QMS $\mathcal{T}$) has a faithful set of normal invariant states , then $\mathcal{F}(\mathcal{T})$ is an algebra. Indeed, if a belongs to $\mathcal{F}(\mathcal{T})$, then $a^*a = \mathcal{T}_t(a^*)\mathcal{T}(a) \le \mathcal{T}_t(a^*a)$, because $\mathcal{T}$ is completely positive. Moreover

$$0 \le \mathrm{tr}\ (\mathcal{T}_t(a^*a) - a^*a) = 0$$

for all normal invariant state ρ. Therefore, since the set of invariant states is also faithful, it follows that $\mathcal{T}_t(a^*a) = a^*a$.

The reduced semigroup $\mathcal{T}^{p_R}$ has good asymptotic properties

Theorem 2.1. *The limit*

$$\mathcal{E}(a) = w^* - \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{T}_s^{p_R}(a)ds$$

exists for all $a \in \mathcal{A}$ and defines a normal $\mathcal{T}$-invariant norm-one projection onto the von Neumann subalgebra $\mathcal{F}(\mathcal{T}^{p_R})$ of $p_R\mathcal{A}p_R$. A normal state ω is $\mathcal{T}$-invariant if and only if $\omega \circ \mathcal{E} = \omega$.

We refer to Frigerio and Verri [21] for the proof of this and the following theorem.

Theorem 2.2. *For a QMS $\mathcal{T}$ on a von Neumann algebra $\mathcal{A}$ the following conditions are equivalent:*

(1) *there exists a normal $\mathcal{T}$-invariant norm-one projection $\mathcal{P}$ of $\mathcal{A}$ onto $\mathcal{F}(\mathcal{T})$,*
(2) $w - \lim_{t\to\infty} t^{-1}\int_0^t \mathcal{T}_{*s}(\varphi)ds$ *exists for all $\varphi \in \mathcal{A}_*$,*
(3) $w^* - \lim_{t\to\infty} t^{-1}\int_0^t \mathcal{T}_s(p_R)ds = \mathbb{1}$.

If the above conditions are satisfied, then, for all $a \in \mathcal{A}$

$$\mathcal{P}(a) = w^* - \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{T}_s(a)ds.$$

The above results are not easily applicable to concrete QMS arising in physical applications. We shall introduce simpler tools further.

3. The transient projection and irreducible QMS

In this section we introduce a projection in $\mathcal{A}$ determined by states with finite sojourn time and call it *transient projection*.

We shall use quadratic forms language and results following the book of Kato [24].

Definition 3.1. *Let x be a positive operator $\mathcal{A}$. The* form-potential *of x is the quadratic form $\mathfrak{U}(x)$ on the domain*

$$D(\mathfrak{U}(x)) = \left\{ u \in \mathsf{h} \mid \sup_{t\ge 0}\int_0^t \langle u, \mathcal{T}_s(x)u\rangle ds < \infty \right\},$$

defined, for all $u \in D(\mathfrak{U}(x))$, by

$$\mathfrak{U}(x)[u] = \int_0^\infty \langle u, \mathcal{T}_s(x)u\rangle ds.$$

Note that $\mathfrak{U}(x)[u]$ can be interpreted as the time spent in x by the state u.

The quadratic form $\mathfrak{U}(x)$ is clearly a symmetric and positive form and by Thm. 3.13a and Lemma 3.14a of [24] it is also closed. Therefore, when it is densely defined, it is represented by a self-adjoint operator (see Th.2.1, p.322, Th. 2.6, p.323 and Th. 2.23 p.331 of [24]). This motivates the following definition.

Definition 3.2. *Let x be a positive operator in $\mathcal{A}$ such that $D(\mathfrak{U}(x))$ is dense. The* potential *of x is the self-adjoint operator $\mathcal{U}(x)$ which represents the quadratic form $\mathfrak{U}(x)$.*

Form-potentials provide in a natural way subharmonic projections.

Theorem 3.1. *Let x be a positive operator in $\mathcal{A}$. The orthogonal projections onto*

i) *the closure of $D(\mathfrak{U}(x))$,*
ii) *the closed subspace $\{u \in \mathsf{h} \mid \mathfrak{U}(x)[u] = 0\}$,*

are subharmonic.

We refer to Fagnola and Rebolledo [18] Prop. 3 p. 292, Prop. 4 p. 294 for the proof.

We denote by $\mathcal{A}_{\text{int}}$ the cone of positive operators $x \in \mathcal{A}$ with $\mathcal{U}(x)$ well defined and bounded on h. For each positive operator x on h we call *support* of x and denote it by $s(x)$ the projection onto $\ker(x)^{\perp}$. Let

$$\mathcal{S} := \{\ p \in \mathcal{A} \mid p^2 = p,\ p = p^*,\ \exists x \in \mathcal{A}_{\text{int}} \text{ s.t. } p = s(\mathcal{U}(x))\ \}$$

and define the projection p_T in $\mathcal{A}$

$$p_T := \vee_{p \in \mathcal{S}}\, p.$$

Definition 3.3. *We call p_T the* transient projection *associated with the QMS $\mathcal{T}$.*

The transient projection is orthogonal to the support of invariant states. Indeed

Proposition 3.1. *We have $p_T \leq p_R^{\perp}$.*

Proof. If $p = s(\mathcal{U}(x))$ with $x \in \mathcal{A}_{\text{int}}$, and ω is a $\mathcal{T}$-invariant state, we have

$$\|\omega\| \cdot \|\mathcal{U}(x)\| \geq \omega(\mathcal{U}(x)) = \int_0^\infty \omega(\mathcal{T}_s(x))ds = \int_0^\infty \omega(x)ds,$$

which implies $\omega(\mathcal{U}(x)) = 0$. Since ω is faithful on the subalgebra $s(\omega)\mathcal{A}s(\omega)$, this implies that $s(\omega)\mathcal{U}(x) = 0$, i.e. $\overline{\mathcal{U}(x)(\mathsf{h})} \subseteq \ker s(\omega)$. It follows that the support $s(\mathcal{U}(x))$ of $\mathcal{U}(x)$ is contained in $\ker p_R$, i.e. we obtain $p \leq p_R^{\perp}$ for all $p \in \mathcal{S}$, which implies $p_T \leq p_R^{\perp}$. □

We define also

Definition 3.4. *The projection* $p_{R_0} = p_R^\perp - p_T$ *is called* null recurrent projection.

By Theorem 3.1 ii) each projection p in $\mathcal{S}$ is superharmonic. It is not clear whether the supremum of a family of superharmonic projection is still superharmonic. However we can prove the following

Theorem 3.2. *Suppose that the Hilbert space* h *is separable. Then the projection* p_T *is superharmonic. Moreover there exists an increasing sequence* $(p_n)_{n\geq 1}$ *of projections in* h *with* $\mathcal{U}(p_n)$ *bounded for each* $n \geq 1$ *and* $p_T = \sup_n p_n$.

We refer to [29] for the proof. As a consequence we have the following

Proposition 3.2. *Suppose that the Hilbert space* h *is separable. The subalgebra* $p_T\mathcal{A}p_T$ *is* $\mathcal{T}_t$ *invariant for all* $t \geq 0$.

Proof. Indeed, for all positive $x \in \mathcal{A}$ with $x = p_T x p_T$, we have $x \leq \|x\| p_T$ and

$$0 \leq p_T^\perp \mathcal{T}_t(x) p_T^\perp \leq \|x\| p_T^\perp \mathcal{T}_t(p_T) p_T^\perp = 0,$$

since $\mathcal{T}_t(p_T) \leq p_T$. It follows then from Lemma 2.1 that $p_T\mathcal{T}_t(x)p_T = \mathcal{T}_t(x)$. □

The asymptotic behaviour of $\mathcal{T}$ on the algebra $p_T\mathcal{A}p_T$ can be described easily. Indeed, let $(p_n)_{n\geq 1}$ be the sequence of projections in h as in Theorem 3.2. For all $n \geq 1$ and all $u \in \mathsf{h}$ the function $t \to \langle u, \mathcal{T}_t(p_n)u\rangle$ is uniformly continuous and integrable on $[0, +\infty[$. Therefore it vanishes for $t \to \infty$. It follows that $\mathcal{T}_t(p_n)$ converges strongly to 0 as t tends to infinity.

As a consequence we have the following

Corollary 3.1. *Suppose that the Hilbert space* h *is separable. Then the restriction of* $\mathcal{T}$ *to* $p_T\mathcal{A}p_T$ *has no normal invariant state.*

Proof. Indeed, if ω is a normal state on $p_T\mathcal{A}p_T$, then $\lim_{n\to\infty}\omega(p_n) = \omega(p_T) = 1$. If ω is also $\mathcal{T}$-invariant,then there exists a $m > 1$ such that

$$1/2 < \omega(p_m) = \omega(\mathcal{T}_t(p_m))$$

for all $t \geq 0$. Since $\mathcal{T}_t(p_m)$ converges strongly to 0 as t tends to infinity, letting t tend to infinity, we find the contradiction $1/2 < 0$. □

The asymptotic behaviour of reduced QMS's through p_R or p_T is completely different. Moreover it can be shown as in Proposition 3.2 that each superharmonic projection p determines a $\mathcal{T}_t$-invariant algebra $p\mathcal{A}p$ and a reduced semigroup (irrespectively of the separability of h). This motivates the following

Definition 3.5. *A QMS is called*

(1) irreducible *if it has no non-trivial (i.e. not* 0 *or* $\mathbb{1}$*) superharmonic projection,*
(2) transient *if* $p_T = \mathbb{1}$,
(3) recurrent *if* $p_T = 0$,
(4) positive recurrent *or* fast recurrent *if* $p_R = \mathbb{1}$,
(5) null recurrent *or* slow recurrent *if* $p_{R_0} = \mathbb{1}$.

The above terminology is borrowed from classical (with $\mathcal{A}$ commutative) theory of Markov semigroups. It is worth noticing here that an irreducible QMS, on a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathsf{h})$ with h separable, is either recurrent or transient.

4. Existence of normal invariant states

The existence normal invariant states is a crucial issue in several models. In this section we show how to prove it by a simple compactness argument. We restrict ourselves to the case $\mathcal{A} = \mathcal{B}(\mathsf{h})$.

In order to find an invariant state a quite natural starting point are again the points of the Cesàro means

$$\frac{1}{t}\int_0^t \mathcal{T}_{*s}(\varphi)ds, \quad t > 0 \tag{2}$$

where φ is a positive trace-one operator.

Since the maps $\mathcal{T}_{*s}$ are positive and trace preserving, the positive operators (2) have trace 1. Therefore, by weak compactness, one can find several limit points ρ which are invariant under the action of the maps $\mathcal{T}_{*t}$. However, there is no reason for tr (ρ) to be equal to 1.

We introduce then the following

Definition 4.1. *A sequence* $(\omega_n)_{n\geq 1}$ *in the Banach space of trace-class operators on* h *is* tight *if, for every* $\varepsilon > 0$*, there exists a finite rank projection* p *and an* $n_0 > 0$ *such that* tr $(\omega_n p) > 1-\varepsilon$ *for every* $n \geq n_0$.

Tightness allows to prove easily the following

Theorem 4.1. *A tight sequence of normal states admits a subsequence converging weakly to a normal state.*

We now give a sufficient tightness condition.

For each self-adjoint operator Y, bounded from below, with spectral resolution $(E_r)_{r\in\mathbb{R}}$, let $Y\wedge r$ be the truncated bounded operator $Y\wedge r = YE_r + rE_r^{\perp}$.

Theorem 4.2. *Let* $\mathcal{T}$ *be a QMS on* $\mathcal{B}(\mathsf{h})$*. Suppose that there exist two self-adjoint operators* X *and* Y *with* X *positive and* Y *bounded from* *below and with finite dimensional spectral projections associated with bounded intervals such that*

$$\int_0^t \langle u, \mathcal{T}_s(Y \wedge r)u\rangle ds \leq \langle u, Xu\rangle \tag{3}$$

for all $t, r \geq 0$ *and all* $u \in \mathrm{Dom}(X)$*. Then the QMS* $\mathcal{T}$ *has a normal invariant state.*

Proof. We follow the proof given in [16].

Let $-b$ $(b > 0)$ be a lower bound for Y. Clearly, for each $r \geq 0$ we have $Y \wedge r \geq -bE_r + rE_r^{\perp} = -(b+r)E_r + r\mathbb{1}$. The inequality (3) yields

$$-(b+r)\int_0^t \langle u, \mathcal{T}_s(E_r)u\rangle ds + rt\|u\|^2 \leq \langle u, Xu\rangle$$

for all $u \in \mathrm{dom}(X)$. We normalize u and denote by $|u\rangle\langle u|$ the pure state associated with the unit vector u i.e. the rank-one projection $|u\rangle\langle u|v = \langle u, v\rangle u$. Dividing by $t(b+r)$, for all $t, r > 0$ we have then

$$\frac{1}{t}\int_0^t \mathrm{tr}\ \left(\mathcal{T}_{*s}(|u\rangle\langle u|)E_r\right) ds \geq \frac{r}{b+r} - \frac{\langle u, Xu\rangle}{t(b+r)}.$$

Therefore, for all $\varepsilon > 0$, there exists $t(\varepsilon) > 0, r(\varepsilon) > 0$ such that

$$\frac{1}{t}\int_0^t \mathrm{tr}\ \left(\mathcal{T}_{*s}(|u\rangle\langle u|)E_{r(\varepsilon)}\right) ds \geq 1 - \varepsilon.$$

The conclusion follows then from Theorem 4.1 and the $\mathcal{T}_{*t}$-invariance of limit points of (2). □

Remark. Once the existence of an invariant state ω is established, if the QMS $\mathcal{T}$ is irreducible, ω is faithful by Proposition 2.1.

5. Convergence to a faithful invariant state

Suppose that a QMS $\mathcal{T}$ admits a faithful normal invariant state ρ and let $\mathcal{P} : \mathcal{A} \to \mathcal{F}(\mathcal{T})$ be the projection as in Theorem 2.1.

Following the physical terminology, we say that $\mathcal{T}$ *approaches the invariant state* if it satisfies

$$w^* - \lim_{t\to\infty} \mathcal{T}_t(a) = \mathcal{P}(a), \tag{4}$$

for each $a \in \mathcal{A}$. This property can be established if we are able to characterise the peripheral spectrum of $\mathcal{T}_t$ (see [2] and the references therein). Clearly, it might not hold even when h is finite dimensional and $\mathcal{T}_t(x) = \mathrm{e}^{itH}x\mathrm{e}^{-itH}$ with H self-adjoint. On the other hand it is often difficult to characterise the peripheral spectrum of $\mathcal{T}_t$.

Frigerio and Verri [21] developed the following method which has the advantage of leading to simple conditions on the generator of the QMS. Let us define

$$\mathcal{N}(\mathcal{T}) = \{ a \in \mathcal{A} \mid \mathcal{T}_t(a^*a) = \mathcal{T}_t(a^*)\mathcal{T}_t(a),\ \mathcal{T}_t(aa^*) = \mathcal{T}_t(a)\mathcal{T}_t(a^*) \} .$$

It is not hard to prove that the following

Proposition 5.1. *$\mathcal{F}(\mathcal{T})$ is a subset of $\mathcal{N}(\mathcal{T})$.*

Proof. Indeed, if a belongs to $\mathcal{N}(\mathcal{T})$, since the maps $\mathcal{T}_t$ are completely positive and identity preserving we have

$$a^*a = \mathcal{T}_t(a^*)\mathcal{T}_t(a) \leq \mathcal{T}_t(a^*a), \qquad aa^* = \mathcal{T}_t(a)\mathcal{T}_t(a^*) \leq \mathcal{T}_t(aa^*).$$

Moreover, ρ is an invariant state and we have

$$0 \leq \mathrm{tr}\ (\rho\,(\mathcal{T}_t(a^*a) - a^*a)) = \mathrm{tr}\ (\rho\,(\mathcal{T}_t(a^*a) - a^*a)) = 0.$$

Since ρ is faithful it follows that $\mathcal{T}_t(a^*a) = \mathcal{T}_t(a^*)\mathcal{T}_t(a)$. In a similar way we check the identity $\mathcal{T}_t(aa^*) = \mathcal{T}_t(a)\mathcal{T}_t(a^*)$ It follows that a belongs to $\mathcal{N}(\mathcal{T})$. □

The following is the sufficient condition (which is also necessary under some additional assumption, see e.g. [15]) due to Frigerio and Verri.

Theorem 5.1. *Let $\mathcal{T}$ be a QMS with a faithful normal invariant state. If $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$ then (4) holds.*

We now proceed to the study of a smaller class of QMS, namely those whose generator can be written in a generalised Lindblad type form.

6. The minimal QMS

We now introduce a class of QMS on $\mathcal{B}(\mathsf{h})$ whose infinitesimal generator is associated with quadratic forms $\pounds(x)$ ($x \in \mathcal{B}(\mathsf{h})$)

$$\pounds(x)[v,u] = \langle Gv, xu\rangle + \sum_{\ell=1}^{\infty}\langle L_\ell v, xL_\ell u\rangle + \langle v, xGu\rangle$$

($v, u \in \mathrm{Dom}(G)$) where the operators G, L_ℓ satisfy the following assumption:

- **H** *the operator G is the generator of a strongly continuous contraction semigroup on h, L_ℓ are operators on h with $\mathrm{Dom}(L_\ell) \supseteq \mathrm{Dom}(G)$, and $\pounds(\mathbb{1}) = 0$, $\mathbb{1}$ being the identity operator on h.*

These QMS arise in the study of irreversible evolutions of quantum open systems (see [1], [3], [4], [23], [27]). The above formula for $\pounds(x)$ generalises (1) to unbounded operators G, L_ℓ.

It is well-known (see e.g. [12] Sect.3, [14] Sect. 3.3) that, given a domain $D \subseteq \mathrm{dom}(G)$, which is a core for G, it is possible to build up a quantum dynamical semigroup, called the *minimal* QDS and denoted $\mathcal{T}^{(\min)}$, satisfying the equation:

$$\langle v, \mathcal{T}_t^{(\min)}(x)u\rangle = \langle v, xu\rangle + \int_0^t \langle v, \pounds(\mathcal{T}_s^{(\min)}(x))u\rangle ds, \tag{5}$$

for $u, v \in D$.

For each positive $x \in \mathcal{B}(\mathsf{h})$, $\mathcal{T}_t^{(\min)}(x)$ is the l.u.b. of the increasing sequence of bounded operators $(\mathcal{T}_t^{(n)}(x))_{t\geq 0}$ on h defined recursively by

$$\begin{aligned} \mathcal{T}_t^{(0)}(x) &= \mathrm{e}^{tG^*} x \mathrm{e}^{tG} \\ \left\langle v, \mathcal{T}_t^{(n+1)}(x)u\right\rangle &= \left\langle \mathrm{e}^{tG} v, x\mathrm{e}^{tG} u\right\rangle \\ &+ \sum_{\ell\geq 1} \int_0^t \left\langle L_\ell \mathrm{e}^{(t-s)G} v, \mathcal{T}_s^{(n)}(x) L_\ell \mathrm{e}^{(t-s)G} u\right\rangle ds \end{aligned} \tag{6}$$

(see e.g. [8] Prop. 2.3, [14] Ch.3 Sect.3 and also [20], [10]). Since $\pounds(\mathbb{1}) = 0$, it is easy to see that $\mathcal{T}_t^{(\min)}(\mathbb{1}) \leq \mathbb{1}$. Equation (5) determines a unique QMS if and only if $\mathcal{T}_t^{(\min)}(\mathbb{1}) = \mathbb{1}$.

The minimal QDS is characterised by the property: for any w^*-continuous family $(\mathcal{T}_t)_{t\geq 0}$ of positive maps on $\mathcal{B}(\mathsf{h})$ satisfying (5) we have $\mathcal{T}_t^{(\min)}(x) \leq \mathcal{T}_t(x)$ for all positive $x \in \mathcal{B}(\mathsf{h})$ and all $t \geq 0$ (see, for instance, [14] Th. 3.21).

Let $\mathcal{T}_*^{(\min)}$ denote the predual semigroup on $\mathcal{I}_1(\mathsf{h})$ with infinitesimal generator $\mathcal{L}_*^{(\min)}$. It is worth noticing here that $\mathcal{T}_*^{(\min)}$ is a *weakly* continuous semigroup on the Banach space of trace class operators on h, hence it is *strongly* continuous. The linear span $\mathcal{V}$ of trace-class operators of the form $|u\rangle\langle v|$ is contained in the domain of $\mathcal{L}_*^{(\min)}$. Thus we can write the equation (5) as follows

$$\mathrm{tr}\ (|u\rangle\langle v|\mathcal{T}_t(x)) = \mathrm{tr}\ (|u\rangle\langle v|x) + \int_0^t \mathrm{tr}\ \left(\mathcal{L}_*^{(\min)}(|u\rangle\langle v|)\mathcal{T}_s(x)\right) ds. \tag{7}$$

It can be shown that the solution to (5), (7) is unique whenever the linear manifold $\mathcal{L}_*^{(\min)}(\mathcal{V})$ is big enough. Indeed, we have the following

Proposition 6.1. *If the assumption* **H** *holds, the following conditions are equivalent:*

(i) *the minimal QDS $\mathcal{T}^{(\min)}$ is Markov,*
(ii) *the linear manifold $\mathcal{V}$ is a core for $\mathcal{L}_*^{(\min)}$,*
(iii) *for each $\lambda > 0$ there exists no $x \in \mathcal{B}(\mathsf{h})$ such that $\pounds(x) = \lambda x$.*

We refer to [12] Th. 3.2 or [14] Prop. 3.32 for the proof. It can be shown also that, if $\mathcal{T}^{(\min)}$ is Markov, then it is the unique QMS satisfying (5).

These are the basic steps for constructing our QMS from a form generator $\pounds$. The above conditions (i),...,(iii), however, are difficult (often impossible) to check in the applications. An easier and applicable sufficient condition based on the existence of a positive self-adjoint operator C satisfying

$$\sum_{\ell\geq 1} L_\ell^* L_\ell \leq C, \qquad \pounds(C) \leq bC,$$

$b > 0$ constant, was given in [8]. We shall not discuss further this problem here.

We now look for a characterisation of subharmonic projections based on the operators G and L_ℓ. Denote by $R(p)$ the range of a projection p on h.

Theorem 6.1. *Suppose that the minimal QDS $\mathcal{T}$ associated with the operators G, L_ℓ satisfying the hypothesis* **H** *is Markov. Let $(P_t)_{t\geq 0}$ be the strongly continuous contraction semigroup on h generated by G. A projection p is subharmonic for $\mathcal{T}$ if and only if*

$$P_t p = p P_t p, \quad L_\ell u = p L_\ell u, \tag{8}$$

for all $u \in$Dom$(G) \cap R(p)$ and all $t \geq 0$, $\ell \geq 1$.

Proof. (Sketch) The QMS $\mathcal{T}$ satisfies the identity (6) without the $(n+1)$ and (n).

Suppose that p is subharmonic, thus $\mathcal{T}_t(p) \geq p$ for all $t \geq 0$. From the identity (6) we obtain $p^\perp \geq \mathcal{T}_t(p^\perp) \geq P_t p^\perp P_t$. Therefore, for all $u \in p$ we have

$$\langle u, P_t p^\perp P_t u\rangle = \|p^\perp P_t u\|^2 = 0,$$

that is $p^\perp P_t p = 0$. Thus $P_t p = p P_t p$, for all $t \geq 0$.

Moreover the equation (5) yields

$$\int_0^t \Big(\langle Gu, \mathcal{T}_s(p^\perp)u\rangle + \sum_{\ell\geq 1}\langle L_\ell u, \mathcal{T}_s(p^\perp) L_\ell u\rangle + \langle u, \mathcal{T}_s(p^\perp) Gu\rangle \Big) ds \leq 0,$$

for all $t \geq 0$ and all $u \in$Dom(G). Differentiating at $t = 0$ we obtain

$$\langle Gu, p^\perp u\rangle + \sum_{\ell\geq 1}\langle L_\ell u, p^\perp L_\ell u\rangle + \langle u, p^\perp Gu\rangle \leq 0.$$

Now, if $pu = u$ the above inequality yields $p^\perp L_\ell u = 0$, i.e. $L_\ell u = p L_\ell u$ for all $\ell \geq 1$ and $u \in$Dom$(G) \cap R(p)$.

The converse can be proved by an induction argument based on the recursive formula (6). Indeed, since $p^\perp P_t p = 0$, we can start from

$$\mathcal{T}_t^{(0)}(p^\perp) = P_t^* p^\perp P_t = p^\perp P_t^* p^\perp P_t p^\perp \leq p^\perp$$

and complete the induction argument (see [17] for the details). □

Condition (8) means that a projection p is subharmonic if and only if its range $R(p)$ is an invariant subspace for all the operators P_t and $\mathrm{Dom}(G) \cap R(p)$ is an invariant subspace for all the operators L_ℓ.

As a consequence we have the following

Corollary 6.1. *Suppose that the minimal QMS associated with the operators G and L_ℓ is Markov. Then it is irreducible if and only if there are no non-trivial invariant subspaces h_0 for all the operators P_t such that $\mathrm{Dom}(G) \cap \mathsf{h}_0$ is L_ℓ-invariant for all $\ell \geq 1$.*

7. Existence of invariant states: conditions on G and L_ℓ's

As we mentioned in the Introduction, however, in the applications usually the operators G, L_ℓ are given. Therefore we now give now a condition involving only these operators.

Definition 7.1. *Given two selfadjoint operators X, Y, with X positive and Y bounded form below, we write $\mathcal{L}(x) \leq -Y$ on D, whenever the inequality*

$$\langle Gu, Xu\rangle + \sum_{\ell=1}^{\infty} \langle X^{1/2} L_\ell u, X^{1/2} L_\ell u\rangle + \langle Xu, Gu\rangle \leq -\langle u, Yu\rangle, \tag{9}$$

holds for all u in a linear manifold D dense in h, contained in the domains of G, X and Y, which is a core for X and G, such that $L_\ell(D) \subseteq D(X^{1/2})$, ($\ell \geq 1$).

This is our sufficient condition based on the operators G, L_ℓ.

Theorem 7.1. *Assume that the hypothesis* **H** *holds and that the minimal QDS $\mathcal{T}$ on $\mathcal{B}(\mathsf{h})$ associated with G, $(L_\ell)_{\ell\geq 1}$ is Markov. Suppose that there exist two self-adjoint operators X and Y, with X positive and Y bounded from below and with finite dimensional spectral projections associated with bounded intervals, such that*

(i) $\mathcal{L}(x) \leq -Y$ *on* D;
(ii) *G is relatively bounded with respect to X;*
(iii) $L_\ell(n+X)^{-1}(D) \subseteq D(X^{1/2})$, *($n, \ell \geq 1$).*

Then QMS $\mathcal{T}$ has a normal invariant state.

Note that the above sufficient conditions always hold for a finite dimensional space h by taking $X = \mathbb{1}$, $Y = 0$ and $D = \mathsf{h}$.

We refer to [16] for the proof. The basic idea is the following formal computation

$$\frac{d}{dt}\left(X - \int_0^t \mathcal{T}_s(Y)ds - \mathcal{T}_t(X)\right) = -\mathcal{T}_t\left(Y + \mathcal{L}(X)\right) \geq 0,$$

by the hypothesis (i). Therefore, since the argument of d/dt vanishes at $t = 0$, it is a positive operator and the inequality (3) follows.

We now turn to the approach to an invariant state. The existence of a faithful normal invariant state allows us to prove easily applicable results on this problem.

We restrict ourselves to the case when the operators G, L_ℓ are bounded referring to [15] Sect. 2 for the general case. The identity $\mathcal{L}(\mathbb{1}) = 0$ (hypothesis **H**) yields $2G = -\sum_\ell L_\ell^* L_\ell + iH$ for a bounded self-adjoint operator H which is easily identified in the applications. Moreover (see [21] Sect. 4) one can prove the

Theorem 7.2. *Let $\mathcal{T}$ be a uniformly continuous QMS and let H, L_ℓ the above elements of $\mathcal{B}(\mathsf{h})$. Then*

$$\mathcal{N}(\mathcal{T}) = \{\, L_1, L_1^*, L_2, L_2^*, \dots \}', \quad \mathcal{F}(\mathcal{T}) = \{\, H, L_1, L_1^*, L_2, L_2^*, \dots \}'.$$

Here $\{\cdots\}'$ denotes the commutator, i.e. the von Neumann algebra of all the elements of $\mathcal{A}$ commuting with the operators listed within braces.

Proof. We check first the identity for $\mathcal{N}(\mathcal{T})$. Let $a \in \mathcal{N}(\mathcal{T})$. Differentiating the identity $\mathcal{T}_t(a^*a) = \mathcal{T}_t(a^*)\mathcal{T}_t(a)$ at $t = 0$ we find $\mathcal{L}(a^*a) = a^*\mathcal{L}(a) + \mathcal{L}(a^*)a$. For any $a \in \mathcal{A}$ a straightforward computation yields

$$\mathcal{L}(a^*a) - a^*\mathcal{L}(a) - \mathcal{L}(a^*)a = \sum_{\ell \geq 1}[L_\ell, a]^*[L_\ell, a].$$

The right-hand side is a sum of positive operators, therefore, if a belongs to $\mathcal{N}(\mathcal{T})$, then a commutes with all the L_ℓ's. In a similar way we can show that also a^* commutes with all the L_ℓ's and, taking the adjoint of commutators $[L_\ell, a], [L_\ell, a^*]$, that a commutes also with all the L_ℓ^*'s.

Conversely, if $[L_\ell, a] = 0 = [L_\ell^*, a]$ for all $\ell \geq 1$, then a simple computation yields $\mathcal{L}(a) = i[H, a]$. This is the infinitesimal generator of the QMS of operators $a \to \mathrm{e}^{itH} a \mathrm{e}^{-itH}$ therefore $\mathcal{T}_t(a) = \mathrm{e}^{itH} a \mathrm{e}^{-itH}$ for all $t \geq 0$. Thus a belongs to $\mathcal{N}(\mathcal{T})$.

Finally, if a belongs to $\mathcal{F}(\mathcal{T})$, then it belongs to $\mathcal{F}(\mathcal{T})$. Therefore it commutes with all the L_ℓ, L_ℓ^*'s. Moreover, by the above argument, we have $\mathcal{T}_t(a) = e^{itH} a e^{-itH}$. The identity $\mathcal{T}_t(a) = a$ for all $t \geq 0$ implies that a commutes also with H.

Conversely, if a commutes with H and all the L_ℓ, L_ℓ^*'s, the $\mathcal{L}(a) = 0$. Thus $\mathcal{T}_t(a) = a$ for all $t \geq 0$. □

Conclusion. The above methods allow us to study the behaviour of the dynamics of a quantum open system given through QMS (master equation). The inspiration from the classical theory of Markov semigroups and processes led to the development of simple, powerful and easily applicable tools. We shall not discuss applications to concrete physical models here for lack of space. The interested reader can find some of them [19].

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